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Recap

$X \in M_n(\mathbb{R}) \rightsquigarrow$ vector field $(\tilde{X}^{(r)}_g)_{g \in GL_n(\mathbb{R})}$ on $GL_n(\mathbb{R})$

by $\tilde{X}^{(r)}_g(f) = \frac{d}{dt} f(g e^{tX})|_{t=0}$

left invariance : $(L_h)_\#(\tilde{X}^{(r)}) = \tilde{X}^{(r)}$

for $L_h : g \mapsto hg$ $\quad ((L_h)_\# : T_g GL_n(\mathbb{R}) \to T_{hg} GL_n(\mathbb{R}))$

algebraic structure on vector fields

M: manifold, $X = (X_p)_{p \in M}$, $Y = (Y_p)_{p \in M}$: vec. flds.

when $f : M \to \mathbb{R}$ is smooth,

$Y(f) : p \mapsto Y_p(f)$ is a smooth func. on M.

$\rightsquigarrow q \mapsto X_q(Y(f))$ also smooth func. on M

problem : for fixed q, $\underset{\text{smooth on } M}{f} \mapsto X_q(Y(f))$ does

not satisfy the Leibniz rule at q.

i.e. $f \mapsto X_q(Y(f))$ is not a tang. vec at q

($f \mapsto X(Y(f))$ is a 2nd ord diff op.)

Prop 1. $[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$

defines a tangent vector at p.

Proof. Step 1 $Y(f_1 f_2) = f_1 Y(f_2) + Y(f_1) f_2$ as funcs.

Step 2 comput. of $X_p(Y(f_1 f_2))$

use the Leibniz rule at p ;

$X_p(Y(f_1)) f_2(p) + Y_p(f_1) X_p(f_2) + X_p(f_1) Y_p(f_2) + f_1(p) X_p(Y(f_2))$

Step 3 Leibniz rule for $[X,Y]_p$

(Step 2) − ($X \leftrightarrow Y$ in Step 2).

cancellation for $Y_p(f_i) X_p(f_j)$

$\rightsquigarrow X_p(Y(f_1)) f_2(p) - Y_p(X(f_1)) f_2(p) + f_1(p) X_p(Y(f_2)) - f_1(p) Y_p(X(f_2))$

$= [X,Y]_p(f_1) f_2(p) + f_1(p) [X,Y]_p(f_2)$

Prop 2. On $GL_n(\mathbb{R})$: $[\tilde{X}^{(r)}, \tilde{Y}^{(r)}] = \tilde{Z}^{(r)}$
for $Z = XY - YX$. $(X, Y \in M_n(\mathbb{R}))$

Proof. Step 1. $\tilde{X}^{(r)}(\tilde{Y}^{(r)}(f_{ij})) = \sum_k (XY)_{kj} f_{ik}$
for the coord. funcs $f_{ij} : g \mapsto g_{ij}$.
$\because$ this follows from $\tilde{Y}(f_{ij}) = \sum_\ell Y_{\ell j} f_{i\ell}$.

Step 2 $[\tilde{X}^{(r)}, \tilde{Y}^{(r)}](f_{ij}) = \tilde{Z}^{(r)}(f_{ij})$
from Step 1.

Step 3 $[\tilde{X}^{(r)}, \tilde{Y}^{(r)}](f) = \tilde{Z}^{(r)}(f)$ for all f.
- use Leibniz rule to get claim for polynomials
  then approximation for smooth funcs.
- tangent vec. only sees the first order
  terms. (F_{ij} in $F(q) = F(p) + \sum F_{ij} \cdot x_{ij} + \cdots$)
  $\uparrow$ around p. $\uparrow$ fixed $\uparrow$ (i,j)-th coord of $q-p$

  * f_{ij} $(i,j = 1, \ldots, n)$ rep. all "directions"

Def. a <u>Lie algebra</u> is given by
- vector space $\mathfrak{g}$
- bilinear map (<u>Lie bracket</u>) $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ $[X, Y]$
  - $[X, Y] = -[Y, X]$
  - $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$
  (<u>Jacobi identity</u>)

Examples
- $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ with $[X, Y] = XY - YX$
  or $\mathfrak{gl}_n(\mathbb{C}), \ldots$
- $\mathfrak{X}(M) = \{$ vector fields on $M \}$

Def. a homomorphism of Lie algs is a
linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ s.t. $f[X, Y] = [fX, fY]$.

Ex. $\mathfrak{gl}_n(\mathbb{R}) \to \mathfrak{X}(GL_n(\mathbb{R}))$, $X \mapsto \tilde{X}^{(r)}$

from Lie groups to Lie algebras

G : Lie group (enough to take $G \subset GL_n(\mathbb{R})$).

goal : make $\mathfrak{g} = T_e G$ a Lie alg. with bracket coming from $\mathfrak{X}(G)$.

how to go from $T_e G$ to $\mathfrak{X}(G)$:

consider $L_h : G \to G$, $g \mapsto hg$ for each h

$X \in T_e G \to (L_h)_\# (X) \in T_h G$

$\rightsquigarrow$ we get a vector field $\tilde{X}^{(r)}$ by $\tilde{X}^{(r)}_h = (L_h)_\# X$

$\rightsquigarrow$ we can try to define $[X, Y]$ by $[\tilde{X}^{(r)}, \tilde{Y}^{(r)}]_e$

Prop 3 $X \mapsto \tilde{X}^{(r)}$ is a linear isom. from $T_e G$ to $\mathfrak{X}(G)^{L_G} = \{$ left invar. vector fields ; $X' = (X'_g)_g$ s.t. $(L_h)_\# X' = X'$ $\}$

Proof Step 1 $\tilde{X}^{(r)}$ is left invar.

$$((L_h)_\# \tilde{X}^{(r)})_g = (L_h)_\# (\tilde{X}_{h^{-1}g}) \underset{\text{def. of } \tilde{X}^{(r)}_g}{=} (L_h)_\# (L_{h^{-1}g})_\# X \underset{\text{functoriality}}{=} (L_g)_\# X$$
$$= \tilde{X}^{(r)}_g .$$

Step 2 $X' = (X'_g)_g$ left inv. $\Rightarrow$ $X' = \tilde{X}^{(r)}$ for $X = X'_e$

Step 3 $X \mapsto \tilde{X}^{(r)}$ bij, with inv. $X' \mapsto X'_e$.

what do we lose by $G \rightsquigarrow \mathfrak{g} = T_e G$?

Examples

1. $\mathbb{R}$ and $\mathbb{T} = \{ e^{it} : t \in \mathbb{R} \}$ have the same Lie alg. $\mathfrak{g} = \mathbb{R}$, $[X, Y] = 0$ commutative bracket

2. $SU(2)$ and $SO(3)$ have the same Lie alg. $\mathfrak{g}$ is 3-dim, basis x_1, x_2, x_3 with $[x_i, x_{i+1}] = x_{i+2}$ index mod 3.

Fact G connected, $\Gamma < G$ discrete subgroup.

then
- Γ is a subgroup of $Z(G)$ (center)
- G/Γ is a Lie group, with same Lie algebra as G.

in fact G_1, G_2 conn. Lie groups

$\mathfrak{g}_1 \simeq \mathfrak{g}_2 \iff$ universal covers $\widetilde{G}_1$ and $\widetilde{G}_2$ are isomorphic.

Exponential map (first part of Lie alg $\rightsquigarrow$ Lie grp)

Goal : make sense of e^{tX} in general.

Rem if $G < GL_n(\mathbb{R})$, $T_eG \subset T_e GL_n(\mathbb{R}) \simeq M_n(\mathbb{R})$.

so given $X \in \mathfrak{g} = T_eG$, e^{tX} makes sense in $GL_n(\mathbb{R})$ Do we stay in G?

A. Yes, but we need some explanation.

Ex. $G = SO(n) \rightsquigarrow$ Lie alg $\mathfrak{so}_n = \{X \in M_n(\mathbb{R}), X^t = -X\}$

$(e^{sX})^t = \left(\sum_{k=0} \frac{s^k}{k!} X^k\right)$