From Lie algebras to Lie groups

Exponential map : want to make sense of e^{tX}

suppose $G < GL_n\mathbb{R}$ (matrix grp)

$\mathfrak{g} = T_eG$ makes sense as a subsp. of $M_n(\mathbb{R})$

(as img of $j_\# : T_eG \to T_e GL_n\mathbb{R}$)

Q. given $X \in \mathfrak{g}$, does $e^{tX} = I_n + tX + \frac{t^2}{2}X^2 + \cdots$

stay in G ?

A. YES, but we need some elaboration

Example $G = SO(n) = \{x \in GL_n(\mathbb{R}),\ x^t = x^{-1},\ \det x = 1\}$

1. $\mathfrak{g}$ is $\mathfrak{so}_n = \{X \in M_n(\mathbb{R}) : X^t = -X\}$.

$\because$ $(I_n + \varepsilon X)^t (I_n + \varepsilon X) = I_n \mod \varepsilon^2$

gives $\varepsilon(X^t + X) = 0. \Rightarrow X^t = -X$

$\det(I_n + \varepsilon X) = 1 + \varepsilon \operatorname{Tr} X + \cdots = 0 \mod \varepsilon^2$

2. $X \in \mathfrak{so}_n \Rightarrow e^{sX} \in SO(n)$. for $s \in \mathbb{R}$

$$(e^{sX})^t = \sum_{s=0}^{\infty} \left(\frac{s^k}{k!} X^k\right)^t = \sum \frac{s^k}{k!} (\underbrace{X^t}_{-X})^k = e^{-sX}$$

so $(e^{sX})^t = (e^{sX})^{-1}$.

$\det(e^{sX}) = e^{\operatorname{Tr} sX} = e^0 = 1$.

Generally (for mfds) $\tilde{X}$: vec. field on M.

$p \in M \rightsquigarrow \exists \varepsilon > 0,\ \exists!\ \varphi : (-\varepsilon, \varepsilon) \to M$ s.t.

$\varphi(0) = p,\ \varphi'(t) = \tilde{X}_{\varphi(t)}$ $\left(\frac{df(\varphi(t))}{dt} = \tilde{X}_{\varphi(t)}(f)\right)$

if $X \in \mathfrak{g}$, $\tilde{X}^{(r)} = ((L_g)_\# X)_g$ makes sense as

a vec. field on $GL_n\mathbb{R}$, $\varphi(t) = e^{tX}$ is a

solution to above eq. ($p = e$)

$\Rightarrow$ by uniqueness $\varphi(t) \in G$.

Baker - Campbell - Hausdorff formula

$e^X e^Y = \exp(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \cdots)$

$\cdots$: terms from $[X, \sim]$ and $[Y, \sim]$.

to be precise : for $X, Y \in M_n(\mathbb{R})$ small so that

$A = I_n - e^X e^Y$ small so that we can subst. it in

$\log(1-a) = \sum_{k=1}^{\infty} (-1)^k \frac{a^k}{k}$

then $Z = \log(I_n - A) = \sum (-1)^k \frac{A^k}{k}$ is

$Z = X + Y + \frac{1}{2}[X,Y] + \cdots$ convergent.

Application : Lem. $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ Lie subalg.

$\exists$ neighborhood $\Delta \subset \mathfrak{g}$ of 0 s.t. $\exp(\Delta)\exp(\Delta) \subset \exp(\mathfrak{g})$

Prop. (8.41) G Lie grp. $\mathfrak{g} = T_e G$, $\mathfrak{h} \subset \mathfrak{g}$ Lie subalg.

then $\exists$ Lie grp H, $H \xrightarrow{j} G$ inj. smooth

s.t. $\mathfrak{h} = j_\#(T_e H)$

Key point : assume $G < GL_n(\mathbb{R})$ for some n.

then H = subgrp of $GL_n(\mathbb{R})$ generated by e^X, $X \in \mathfrak{h}$.

1) Lem $\Rightarrow \exists$ neigh. $\Delta \subset \mathfrak{gl}_n(\mathbb{R})$ of 0 s.t.

$(\exp(\Delta \cap \mathfrak{h}) \cdot \exp(\Delta \cap \mathfrak{h})) \cap \exp(\Delta) \subset \exp(\Delta \cap \mathfrak{h})$

2) H is a Lie grp. with neighborhood of $h \in H$

- coord from $\Delta \cap \mathfrak{h} \subset \mathfrak{h}$ (open in $\mathbb{R}^k \simeq \mathfrak{h}$)

- coord map $X \mapsto \exp(X) \cdot h$ ◻

Thm. G, H Lie grps $\mathfrak{g}$, $\mathfrak{h}$ their Lie alg's.

G simply connected.

$\alpha : \mathfrak{g} \to \mathfrak{h}$ Lie alg hom

Then $\exists$ $f : G \to H$ Lie grp hom, $f_\# = \alpha$.

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Idea: 0) $f: G \to H$ hom $\Leftrightarrow$ $\Gamma_f = \{(g, f(g)) : g \in G\} \subset G \times H$ subgrp.

same with Lie alg. homs.

1) From $\alpha: \mathfrak{g} \to \mathfrak{h}$ hom $\rightsquigarrow$ get subalg $\mathfrak{k}_0 = \{(X, \alpha(X))\} \subset \mathfrak{g} \oplus \mathfrak{h}$.

$\underset{\text{Prop.}}{\Rightarrow}$ $\exists$ Lie grp K, inj. hom $K \xrightarrow{\tilde{f}} G \times H$ with $\tilde{f}_\#(\mathfrak{k}) = \mathfrak{k}_0$

2) $K \simeq G$ induced by $(\text{proj to } G) \circ \tilde{f}$.

Corresp. map is Lie alg iso $\Rightarrow$ local homeo.

G simply connected $\Rightarrow$ must be homeo.

3) f is given by $(\text{proj to } H) \circ \tilde{f}$ up to $K \simeq G$,

Conseq. if G, G' have isom. Lie algs & G simply conn. $\exists$ $G \xrightarrow{f} G'$ hom, local homeo.

(so $G' = G / \ker f$).

conversely $\Gamma \triangleleft G$ discrete, G conn.

($\Gamma \triangleleft Z(G)$ by looking at $\underset{\text{conn.}}{\mathrm{Ad}_G(\gamma)} \subset \underset{\text{discr.}}{\Gamma}$)

G/Γ has same Lie alg as G.

Rem. more univ. approach to BCH formula

$H = \mathbb{C}\langle X, Y \rangle$ ring of noncomm. polynoms

Hopf algebra by $\Delta(X) = X \otimes 1 + 1 \otimes X$, $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$

<u>free Lie algebra</u> on two generators

$\mathfrak{f}_2 = \{Z \in H : \Delta(Z) = Z \otimes 1 + 1 \otimes Z\} \ni X, Y, XY - YX, \ldots$

e^X, e^Y make sense in the <u>degree completion</u> of H

and $Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \cdots$

and we have $e^X e^Y = e^Z$, $Z \in \hat{\mathfrak{f}}_2$

Def. a finite dimensional rep. of $\mathfrak{g}$ is given by

- fin. dim vec. sp. V $(\simeq \mathbb{R}^n$ or $\mathbb{C}^n)$
- Lie algebra hom. $\mathfrak{g} \xrightarrow{\pi} \mathfrak{gl}(V) = \{ T \in \mathrm{End}(V) \}$

$[T, S] = TS - ST$

i.e. collection of lin. transforms. π_X for $X \in \mathfrak{g}$

$\pi_{[X,Y]} = [\pi_X, \pi_Y]$

Examples

1 adjoint representation : $V = \mathfrak{g}$, $\pi_X Y = [X, Y]$

$\pi_{[X,Y]} Z = [\pi_X, \pi_Y] Z \iff$ Jacobi identity.

2 $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$, $V_n = \langle x^n, x^{n-1}y, \ldots, y^n \rangle_{\mathbb{C}\text{-span}}$

$(n+1)$-dim

$\pi^n_H = x\partial_x - y\partial_y$, $\pi^n_E = x\partial_y$, $\pi^n_F = y\partial_x$

for $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in \mathfrak{sl}_2$

i.e. $\pi^n_H \, x^{n-k} y^k = (n-2k) x^{n-k} y^k$ ...

Rem. G simply connected (real) Lie grp.

rep. of $\mathfrak{g}$ $\underset{\text{Th'm}}{\equiv}$ rep of G.

without simpl. conn. rep of $\mathfrak{g}$ $\equiv$ rep of univ. cov.