# Representation theory of $\mathfrak{sl}_2$.

Recall :

$$\mathfrak{sl}_2(\mathbb{R}) = \{ X \in \mathfrak{gl}_2(\mathbb{C}) = M_2(\mathbb{C}) \ ; \ \mathrm{Tr}\, X = 0 \}$$

$$= \left\langle H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\rangle$$

$$V_n = \langle x^n, x^{n-1} y, \cdots, y^n \rangle_{\mathbb{R}\text{-span}}$$

$$\pi^n_H = x\partial_x - y\partial_y |_{V_n}, \quad \pi^n_E = x\partial_y |_{V_n}, \quad \pi^n_F = y\partial_x |_{V_n}$$

Note : these make sense over , or other fields.

Thm $(\pi^n, V_n)$ exhaust irreducible finite dim. reps of $\mathfrak{sl}_2(\mathbb{C})$.

- $(\pi, V)$ irred $\overset{\text{def}}{\Leftrightarrow}$ $0$ & $V$ are the only $\pi$-inv subsp.
- the claim holds over any field of char $0$.

Key points

- $(\pi, V)$ rep. of $\mathfrak{sl}_2$ $\rightsquigarrow$ eigendecomp for $\pi_H$.

$$V = \bigoplus_\lambda V_\lambda \qquad \pi_H = \lambda \text{ on } V_\lambda$$

- $E V_\lambda = V_{\lambda+2}$ from $[H, E] = 2E$
  $F V_\lambda = V_{\lambda-2}$ from $[H, F] = -2F$.
- $[E, F] = H$ forces $\lambda$ to be integer. - -

Proof of $(\pi, V)$ irred $\Rightarrow \exists\, n$ $(\pi, V) \simeq (\pi^n, V_n)$

we will write $\pi_H v = Hv$ for $v \in V$, etc.

Step 0 put $V_\lambda^{(k)} = \{ v \in V : (H-\lambda)^k v = 0 \}$

so $V_\lambda^{(1)} \subset V_\lambda^{(2)} \subset \cdots$, $V = \bigoplus_\lambda V_\lambda^{(k)}$ for big $k$.

Put $V_\lambda = V_\lambda^{(k)}$ for big enough $k$

Step 1 $E V_\lambda^{(k)} \subset V_{\lambda+2}^{(k)}$, $F V_\lambda^{(k)} \subset V_{\lambda-2}^{(k)}$

$[H, E] = 2E \Rightarrow E(H-\lambda) = (H-(\lambda+2))E$

$\Rightarrow E(H-\lambda)^k = (H-(\lambda+2))^k E$ by induction

Step 2. when $(0\neq)\, v \in V_\lambda^{(1)} \cap \ker E$

$W = \langle v, Fv, F^2 v, \cdots \rangle$ is $\mathfrak{sl}_2(\mathbb{C})$-invariant.

$F$-invariance is obvious.

$H$-invariance: $F^k v \in V^{(1)}_{\lambda - 2k}$ from Step 1

so $H$ acts diagonally.

$E$-invariance: $[E,F] = H \Rightarrow EF^k = HF^{k-1} + FEF^{k-1}$

induction on $k$ gives $EF^k v \in \langle v, \cdots, F^{k-1} v \rangle$

Step 3 $\exists\, \lambda$ s.t. $V_\lambda^{(1)} \neq 0$ (highest weight of $\pi$)

take the biggest $\lambda$ that appears in $V = \bigoplus_\lambda V_\lambda$

in real part

Step 1 $\Rightarrow E V_\lambda = 0$.

$(\pi, V)$ irred $\Rightarrow W = V$

Step 4 $\lambda$ is a nonneg. integer. $(= n$ s.t. $(\pi, V) \simeq (\pi^n, V_n))$

$[E,F] = H \Rightarrow \mathrm{Tr}_V H = \mathrm{Tr}_V (EF - FE) = 0$

$H$ has eigenvals $\lambda, \lambda - 2, \cdots, \lambda - 2k$ on $V$

with $k$ s.t. $F^k v \neq 0$ and $F^{k+1} v = 0$

$\mathrm{Tr}_V H = (k+1)\lambda - k(k+1) \Rightarrow \lambda = k$ ◻

Rem extra work for other fields: $H$ is diagonalizable on $V$. (§C.2 in F-H).

Rem. commutative Lie algs can have bad rep. theory

$\mathfrak{g} = \mathbb{C} \Rightarrow \mathfrak{g}$-rep on $V \equiv$ specify $T \in \mathrm{End}(V)$

so indecomposable $\mathfrak{g}$-reps $\leftrightarrow \left( \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \right)^k$

for $\lambda \in \mathbb{C}$, $k = 1, 2, \cdots$

irred. iff $k = 1$

$\mathfrak{g} = \mathbb{C}^2 \Rightarrow \mathfrak{g}$-rep on $V \equiv S, T \in \mathrm{End}(V)$ s.t. $ST = TS$

Structure of Lie algebras.

Goal: understand dichotomy between

- semisimple: very noncommutative; rigid
  $\mathfrak{sl}_n$ $(n \geq 2)$, $\mathfrak{so}_n$ $(n \geq 3)$, ...
- solvable: close to commutative; soft
  comm. $\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{C}\right\}$, ...
  $$\left[\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & ab' - a'b \\ 0 & 0 \end{bmatrix}.$$

and how they "compose" in general Lie algs.

$\mathfrak{g}$ : Lie alg (over $\mathbb{R}$, $\mathbb{C}$, ...)

Def. an <u>ideal</u> of $\mathfrak{g}$ is a subsp. $\mathfrak{h} \subset \mathfrak{g}$ s.t.
$\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$ : $[X, Y] \in \mathfrak{h}$ (write $\mathfrak{h} \triangleleft \mathfrak{g}$)

Ex. $\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right\} \triangleleft \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right\}$

Def the <u>derived</u> <u>subalgebra</u> (commutator subalg)
$\mathcal{D}(\mathfrak{g}) = (\text{span of } [X, Y] \text{ for } X, Y \in \mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$

Rem. $\mathcal{D}(\mathfrak{g}) \triangleleft \mathfrak{g}$ by Jacobi id.
$\mathfrak{g}/\mathcal{D}(\mathfrak{g})$ is commutative.

Def. the <u>lower</u> <u>central</u> <u>seris</u> of $\mathfrak{g}$ : $(\mathcal{D}_n(\mathfrak{g}))_{n=1}^{\infty}$
$\mathcal{D}_1(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})$, $\mathcal{D}_{n+1}(\mathfrak{g}) = [\mathcal{D}_n(\mathfrak{g}), \mathfrak{g}]$.

the <u>derived</u> <u>series</u> of $\mathfrak{g}$ : $(\mathcal{D}^n(\mathfrak{g}))_{n=1}^{\infty}$
$\mathcal{D}^1(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})$, $\mathcal{D}^{n+1}(\mathfrak{g}) = [\mathcal{D}^n(\mathfrak{g}), \mathcal{D}^n(\mathfrak{g})]$

Prop. $\mathcal{D}_n(\mathfrak{g})$, $\mathcal{D}^n(\mathfrak{g}) \triangleleft \mathfrak{g}$.

Proof for $\mathcal{D}_n(\mathfrak{g})$ : suppose we know $\mathcal{D}_{n-1}(\mathfrak{g}) \triangleleft \mathfrak{g}$.
take $X, Z \in \mathfrak{g}$, $Y \in \mathcal{D}_{n-1}(\mathfrak{g})$
we want $[X, \underbrace{[Y, Z]}_{\text{generic elem. in } \mathcal{D}_n(\mathfrak{g})}] \in \mathcal{D}_n(\mathfrak{g})$

(cont.) Jacobi & antisym. rel give

$$[X,[Y,Z]] = \underbrace{[[Z,X],Y]}_{\text{in } \mathcal{D}_n} + \underbrace{[\underbrace{[X,Y]}_{\text{in } \mathcal{D}_{n-1}},Z]}_{\text{in } \mathcal{D}_n} \in \mathcal{D}_n(\mathfrak{g})$$

Rem. we set $\mathcal{D}_{n+1} \subset \mathcal{D}_n$, $\mathcal{D}^{n+1} \subset \mathcal{D}^n$.

Def. $\mathfrak{g}$ is <u>solvable</u> if $\mathcal{D}^k(\mathfrak{g}) = 0$ for $k \gg 1$

<u>nilpotent</u> if $\mathcal{D}_k(\mathfrak{g}) = 0$ for $k \gg 1$

$\mathcal{D}^n \subset \mathcal{D}_n \rightsquigarrow$ nilpot. $\Rightarrow$ solvable

Ex. 1. $\mathfrak{g} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\}$

$\mathcal{D}_1(\mathfrak{g}) = \mathcal{D}^1(\mathfrak{g}) = \mathcal{D}(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in \mathbb{C} \right\}$

$\mathcal{D}^2(\mathfrak{g}) = \mathcal{D}(\mathcal{D}(\mathfrak{g})) = 0$, $\mathcal{D}_2(\mathfrak{g}) = [\mathfrak{g}, \mathcal{D}(\mathfrak{g})] = \mathcal{D}(\mathfrak{g})$

So $\mathfrak{g}$ is solvable but <u>not</u> nilpotent.

2. $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ $\mathcal{D}(\mathfrak{g}) = \mathfrak{g}$ so $\mathfrak{g}$ is not solvable

(in fact $\mathfrak{g}$ is <u>simple</u> : no nontriv ideal)

Prop. $\mathfrak{g}$ is solvable iff $\exists$ dec. seq. $\mathfrak{g} \triangleright \mathfrak{g}_1 \triangleright \mathfrak{g}_2 \triangleright \cdots$ of subalgs s.t. $\left( \begin{array}{l} \mathfrak{g}_k = 0 \text{ for } k \gg 1 \\ \mathfrak{g}_k / \mathfrak{g}_{k+1} \text{ comm.} \end{array} \right.$

Proof $\Rightarrow$ : take $\mathfrak{g}_k = \mathcal{D}^k(\mathfrak{g})$

$\Leftarrow$ : $\mathfrak{g}_k / \mathfrak{g}_{k+1}$ comm $\Rightarrow \mathcal{D}(\mathfrak{g}_k) \subset \mathfrak{g}_{k+1}$

by induction $\mathcal{D}^k(\mathfrak{g}) \subset \mathfrak{g}_k$

so $\mathcal{D}^k(\mathfrak{g}) = 0$ for $k \gg 1$.