Prop. suppose $\mathfrak{h} \triangleleft \mathfrak{g}$ (ideal)

Then $\mathfrak{g}/\mathfrak{h}$, $\mathfrak{h}$ solvable $\Leftrightarrow$ $\mathfrak{g}$ solvable

Key point: $\mathfrak{g}$ solvable $\Leftrightarrow \exists\ \mathfrak{g} = \mathfrak{g}_1 \triangleright \mathfrak{g}_2 \triangleright \cdots \triangleright \mathfrak{g}_k = 0$

$\mathfrak{g}_n/\mathfrak{g}_{n+1}$ comm.

Proof $\Leftarrow$: with $(\mathfrak{g}_n)_n$ as above, put

$\mathfrak{h}_n = \mathfrak{h} \cap \mathfrak{g}_n$, $(\mathfrak{g}/\mathfrak{h})_n =$ img of $\mathfrak{g}_n$ in $\mathfrak{g}/\mathfrak{h}$.

then $\mathfrak{h} = \mathfrak{h}_1 \triangleright \mathfrak{h}_2 \triangleright \cdots$ with same prop.

$\mathfrak{g}/\mathfrak{h} = (\mathfrak{g}/\mathfrak{h})_1 \triangleright (\mathfrak{g}/\mathfrak{h})_2 \triangleright \cdots$

$\Rightarrow$: take $(\mathfrak{h}_n)_{n=1}^k$, $((\mathfrak{g}/\mathfrak{h})_n)_{n=1}^{\ell}$ as above.

$$\mathfrak{g}_n = \begin{cases} \text{inv. img of } (\mathfrak{g}/\mathfrak{h})_n & (\text{for } n < \ell) \\ \mathfrak{h}_{n-\ell} & (n \geq \ell) \end{cases}$$

Then $\mathfrak{g} = \mathfrak{g}_1 \triangleright \mathfrak{g}_2 \triangleright \cdots \triangleright \mathfrak{g}_\ell = \mathfrak{h} \triangleright \mathfrak{g}_{\ell+1} \triangleright \cdots$

$$\mathfrak{g}_n/\mathfrak{g}_{n+1} \cong \begin{cases} (\mathfrak{g}/\mathfrak{h})_n / (\mathfrak{g}/\mathfrak{h})_{n+1} & \text{for } n < \ell \\ \mathfrak{h}_{n-\ell}/\mathfrak{h}_{n-\ell+1} & \text{for } n \geq \ell \end{cases}$$

Cor. $\mathfrak{h}_1, \mathfrak{h}_2 \triangleleft \mathfrak{g}$ solvable ideals

$\Rightarrow \mathfrak{h}_1 + \mathfrak{h}_2$ is a solvable ideal of $\mathfrak{g}$

$\because$ take $\mathfrak{h}_1 \triangleleft (\mathfrak{h}_1 + \mathfrak{h}_2)$, quot $\mathfrak{h}_2/\mathfrak{h}_1 \cap \mathfrak{h}_2$

Prop / Def. $\exists!$ maximal solvable ideal of $\mathfrak{g}$, called the <u>radical</u> of $\mathfrak{g}$; write $\mathrm{Rad}(\mathfrak{g})$.

Def. $\mathfrak{g}$ is <u>semisimple</u> if $\mathrm{Rad}(\mathfrak{g}) = 0$.

<u>simple</u> if $\dim \mathfrak{g} > 1$ and there is no nontriv. ideals of $\mathfrak{g}$

Rem. we don't want $\mathbb{R}$, $\mathbb{C}$ as simple Lie algs.

then simple $\Rightarrow$ semisimple.

Rem. in gen. $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple.

$\mathrm{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ extension

Example $\mathbb{R} \underset{\text{as scalar}}{\to} \mathfrak{gl}_n(\mathbb{R}) \to \mathfrak{sl}_n(\mathbb{R})$.

key : $[e_{ij}, e_{k\ell}] = \delta_{jk} e_{i\ell} - \delta_{i\ell} e_{kj}$

Structure of solvable algebras

Engel's th'm (9.9) V : fin dim vec. sp.

$\mathfrak{g}$: Lie subalg of $\mathfrak{gl}(V)$ s.t. $\forall X \in \mathfrak{g}$ is a nilpotent endom. of V; $\exists n$ $X^n = 0$

then $\exists\, 0 \neq v \in V$ $\forall X \in \mathfrak{g}$ $Xv = 0$

Cor 1. $\exists$ basis $v_1, \dots, v_n < V$ s.t. $\underbrace{\mathfrak{g}}_{\text{mat pres. of } \mathfrak{g}} \subset \left\{ \begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix} \right\}$

Proof of Cor 1. induction on $N = \dim V$

Take $v_1 = v$ as in Th'm, put $V' = V/\langle v_1 \rangle$

$\langle v_1 \rangle$ is $\mathfrak{g}$-inv $\Rightarrow \mathfrak{g}$ acts on V' (by nilpot endos)

$\underset{\text{ind. hypo.}}{\Longrightarrow}$ $\exists$ basis $\bar{v}_2, \dots, \bar{v}_N$ of V' s.t.

$X \bar{v}_k \in \langle \bar{v}_2, \dots, \bar{v}_{k-1} \rangle$ for $2 \leq k \leq N$

$\rightsquigarrow$ take inv. img v_k of $\bar{v}_k$; $(v_1, \dots, v_N)$ will do

Proof of Th'm

Step 1 $\mathrm{ad}_X(T) = [X, T]$ for $T \in \mathrm{End}(V)$

$\rightsquigarrow$ rep of $\mathfrak{g}$ on $\mathrm{End}(V)$.

ad_X is nilpot. on $\mathrm{End}(V)$

$\because X^k = 0 \Rightarrow (\mathrm{ad}_X)^{2k}(T) = X^{2k} T + 2k X^{2k-1} T X + \cdots = 0$

We will prove the claim by induction on $m = \dim \mathfrak{g}$

Step 2 $\exists\, \mathfrak{h} \triangleleft \mathfrak{g}$ s.t. $\dim \mathfrak{h} = m - 1$

$\because$ Take any proper maximal subalg as $\mathfrak{h}$.

$\mathfrak{h}$ is ad_X-inv. for $X \in \mathfrak{h}$ $\Rightarrow \mathfrak{h}$ acts on $\mathfrak{g}/\mathfrak{h}$.

This is by nilpot maps (Step 1) $(\mathrm{ad}_X)_{X \in \mathfrak{h}}$

(cont.) By the induction hypothesis

$\exists\, 0 \neq \bar{Y} \in \mathfrak{g}/\mathfrak{h}_0$ s.t. $\overline{\mathrm{ad}}_X(\bar{Y}) = 0$ for $X \in \mathfrak{h}_0$.

this means $\exists\, Y \in \mathfrak{g} \setminus \mathfrak{h}_0$ $\forall X \in \mathfrak{h}_0$ $[X, Y] \in \mathfrak{h}_0$

So $\mathfrak{h}_0' = \mathfrak{h}_0 + \langle Y \rangle$ is also a subalg of $\mathfrak{g}$

and $\mathfrak{h}_0 \triangleleft \mathfrak{h}_0'$

By maximality $\mathfrak{h}_0' = \mathfrak{g}$ i.e. $\dim \mathfrak{h}_0 = m - 1$.

Take $\mathfrak{h}_0$, $Y \in \mathfrak{g} \setminus \mathfrak{h}_0$ as above., put

$W = \{ v \in V : \forall X \in \mathfrak{h}_0 \quad Xv = 0 \}$ ($\neq 0$ by ind. hyp.)

Step 3 $YW \subset W$.

$\because$ for $X \in \mathfrak{h}_0$, $XYv = Y \underset{\overset{\shortparallel}{0}}{Xv} + \underbrace{[X, Y]}_{\in\, \mathfrak{h}_0} v = 0$

Step 4 find $v \neq 0$ as in claim.

$\because$ enough to find $0 \neq v \in W$ $Yv = 0$

$Y^k = 0$ by assumption. $W \supset YW \supset Y^2 W \supset \cdots = 0$

any $0 \neq v$ from the "last step" will do ∎

Cor 2. $\mathfrak{g}$ is nilpot. $\Leftrightarrow$ $\mathrm{ad} : \mathfrak{g} \curvearrowright \mathfrak{g}$ is by nilpot. maps

$[X_1, [X_2, \cdots (X_{k-1}, X_k]]] = 0$ $\quad [X, (X, \cdots [X, Y]]] = 0$

Proof $\Leftarrow$: Engel's thm gives ideals (of $\mathfrak{g}$)

$\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_m \supset 0$; $\mathfrak{g}_n = \langle v_1, \cdots, v_{m-n+1} \rangle$

$\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}_k) \subset \mathfrak{g}_{k+1}$

Then we have $\mathcal{D}_k(\mathfrak{g}) \subset \mathfrak{g}_k \to 0$

Lie's thm (9.11). V : fin. dim cplx vec. sp.

$\mathfrak{g} \subset \mathfrak{gl}(V)$ solvable Lie alg.

Cor. $\exists$ basis $v_1, \cdots, v_n \subset V$ s.t.

mat. pres. of $\mathfrak{g}$ $\subset \left\{ \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\}$.

Proof by ind. on $m = \dim \mathfrak{g}$.

Step 1. $\exists\, \mathfrak{h} \triangleleft \mathfrak{g}$ $\dim \mathfrak{h} = m-1$

$\because$ $\mathcal{D}(\mathfrak{g}) \neq \mathfrak{g}$ by $\mathcal{D}^k(\mathfrak{g}) \to 0$.

$\mathfrak{g}/\mathcal{D}(\mathfrak{g})$ is comm. $\Rightarrow$ any subsp. is ideal.

take codim 1 subsp., $\mathfrak{h}$ = inv. img

By ind. hyp. $\exists\, v \neq 0$ s.t. $\forall X \in \mathfrak{h}$ $Xv = \lambda(X) v$ for $\lambda(X) \in \mathbb{C}$

Set $W = \{ v' \in V : \forall X \in \mathfrak{h} \;\; Xv' = \lambda(X) v' \}$

same factor as v

Fix $Y \in \mathfrak{g} \setminus \mathfrak{h}$; we want to find an eigenvec. of Y in W; it's enough to show $YW \subset W$.

Step 2 $YW \subset W \Leftrightarrow \lambda([X,Y]) = 0$ for $X \in \mathfrak{h}$.

$\because$ $\underbrace{XYv'}_{\text{want this to be } \lambda(X) Yv'} = \underbrace{YXv'}_{=\lambda(X)Yv'} + \underbrace{[X,Y]v'}_{=\lambda([X,Y])v'}$

Step 3 For $m \in W$ put $U_m = \langle m, Ym, Y^2 m, \cdots \rangle$

$\forall X \in \mathfrak{h}$ $XU_m \subset U_m$, with $\lambda(X)$ as the only eigenval.

$\because$ $XY^k m = \lambda(X) Y^k m + m'$, $m' \in \langle m, Ym, \ldots, Y^{k-1} m \rangle$

by induction on k; $XY^k m = YXY^{k-1} m + \underbrace{[X,Y]}_{\text{in } \mathfrak{h}} Y^{k-1} m$

Step 4 $\lambda([X,Y]) = 0$.

$\because$ $\lambda([X,Y])$ is the only eigenval of $[X,Y] \in \mathfrak{h}$ on U_m

But $\mathrm{Tr}_{U_m}([X,Y]) = 0$ from $XU_m, YU_m \subset U_m$.

$(\dim U_m)\, \lambda([X,Y]) = \mathrm{Tr}_{U_m}([X,Y])$.

$\square$