Exercise set 2.

Prob. 1.

$SU(n) = \{ U \in GL_n(\mathbb{C}) : \bar{U}^t = U^{-1},\ \det U = 1 \}$

⇝ its Lie alg is

$su_n = \{ X \in M_n(\mathbb{C}) : \bar{X}^t = -X,\ \operatorname{Tr} X = 0 \}$.

Treat $SU(n)$ as a subgrp of $GL_n(\mathbb{C})$

⇝ so $T_e SU(n)$ is a subsp. of $T_e GL_n(\mathbb{C}) \cong M_n(\mathbb{C})$

$X \in M_n(\mathbb{C})$ belongs to this subsp.

$\Longleftrightarrow$ $\varphi(s) = e^{sX}$ $(s \in \mathbb{R})$ is in $SU(n)$

To get $\bar{X}^t = -X$: look at $\overline{(e^{sX})}^t = (e^{sX})^{-1} = e^{-sX}$.

- $\overline{(e^{sX})}^t = e^{s\bar{X}^t}$ from $e^{sX} = \sum_{k=0}^{\infty} \frac{s^k}{k!} X^k$, $\overline{X^k}^t = (\bar{X}^t)^k$.
- differentiate $\overline{e^{sX}}^t e^{sX} = I_n$ at $s = 0$

  left hand side gives $\bar{X}^t + X$.

  right hand side gives 0

Conversely $\bar{X}^t = -X \Rightarrow \overline{e^{sX}}^t = e^{-sX}$.

To get $\operatorname{Tr} X = 0$: look at $\det e^{sX} = e^{s \operatorname{Tr} X}$.

Prob 2. Concrete isom $su_2 \cong so_3$.

$su_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C}) : \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} = - \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ a + d = 0 \right\}$

⇝ basis $A_1 = \begin{bmatrix} i & \\ & -i \end{bmatrix}$, $A_2 = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$, $A_3 = \begin{bmatrix} & i \\ i & \end{bmatrix}$

$so_3 = \{ B \in M_3(\mathbb{R}) : B^t = -B \}$

⇝ basis $B_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

(generates rotation in : xy-plane, yz-plane, zx-plane.

computing bracket

$[A_1, A_2] = -2A_3$, $[A_2, A_3] = -2A_1$, $[A_3, A_1] = -2A_2$

$[B_1, B_2] = B_3$, $[B_2, B_3] = B_1$, $[B_3, B_1] = B_2$

then $\tilde{A}_i = -\frac{1}{2}A_i$ satisfy the same rel. as B_i

e.g. $[\tilde{A}_1, \tilde{A}_2] = \frac{1}{4}[A_1, A_2] = -\frac{1}{2}A_3 = \tilde{A}_3$

So $f: su_2 \to so_3$, $A_i \mapsto -2B_i$ is a Lie alg iso.

Prob 3.

1. $\exp : \mathfrak{gl}_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is surjective.

Step 1. $A_t = \begin{bmatrix} 1 & t & & 0 \\ & 1 & \ddots & \\ & & \ddots & t \\ 0 & & & 1 \end{bmatrix}$ is in the img of exp.

$I - A_t$ is nilpot, hence $\log A_t = \log(I_n - (I_n - A_t))$

makes sese by $\log(1-b) = -\sum_{k=1}^{\infty} \frac{b^k}{k}$, $b \rightsquigarrow I_n - A_t$

Step 2 $J_\lambda^{(n)} = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ is in the img of exp.
$(\lambda \neq 0)$

$J_\lambda = \lambda I_n \cdot A_{\frac{1}{\lambda}}$ if $\mu \in \mathbb{C}$, $X \in M_n\mathbb{C}$ are s.t.

$e^\mu = \lambda$, $\exp(X) = A_{\frac{1}{\lambda}}$. $\exp(\mu I_n + X) \underset{\substack{\uparrow \\ \lambda I_n \text{ and } X \text{ comm.}}}{=} J_\lambda$

Step 3 any $A \in GL_n(\mathbb{C})$ is in the img of exp.

$A = Y \begin{bmatrix} J_{\lambda_1}^{(m_1)} & & \\ & \ddots & \\ & & J_{\lambda_k}^{(m_k)} \end{bmatrix} Y^{-1}$ for $Y \in GL_n(\mathbb{C})$
$m_1 + \cdots + m_k = n$, $\lambda_i \neq 0$

If $X_i \in M_{m_i}(\mathbb{C})$ satisfies $\exp(X_i) = J_{\lambda_i}^{(m_i)}$

$\exp\left(\begin{pmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_k \end{pmatrix}\right) = \begin{pmatrix} J_{\lambda_1}^{(m_1)} & & \\ & \ddots & \\ & & J_{\lambda_k}^{(m_k)} \end{pmatrix}$

$\exp\left(Y \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_k \end{bmatrix} Y^{-1}\right) = A$.

2. $\exp : \mathfrak{gl}_n(\mathbb{R}) \to GL_n^+(\mathbb{R})$ is not surjective $(n \geq 2)$

obstruction : $\lambda_1, \dots, \lambda_n$ (cplx) eigenvals of X

$\Rightarrow e^{\lambda_1}, \dots, e^{\lambda_n}$ (cplx) eigenvals of e^X.

$X \in M_n(\mathbb{R}) \Rightarrow$ nonreal eigenvals must come in conjug pairs.

$$A = \begin{bmatrix} -2 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & \ddots \ 1 \end{bmatrix}$$ won't be in the img of exp.

3. $\exp : \mathfrak{sl}_2(\mathbb{C}) \to SL_2(\mathbb{C})$ not surj

$X \in \mathfrak{sl}_2(\mathbb{C})$ has Jordan normal form

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$.

first case : e^X has eigenval 1.
Second : e^X is Diagonalizable, w/ eigenvals e^{λ} & $e^{-\lambda}$.

So $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ eigenval -1, not Diagonalizable

is not in the img.

Prob 4. $ad_X(Y) = \frac{d}{dt} Ad_{\exp(tX)}(Y)\Big|_{t=0}$

agrees with $[X,Y]$ $X, Y \in T_eG$.

recall $[X,Y] = [\tilde{X}^{(r)}, \tilde{Y}^{(r)}]_e$. for left inv. vec. fields $\tilde{X}^{(r)}, \tilde{Y}^{(r)}$

$\tilde{Y}^{(r)}_g(f) = \frac{d}{ds} f(g e^{sY})\Big|_{s=0}$.

$\tilde{X}^{(r)}_g(\tilde{Y}^{(r)}(f)) = \frac{d}{dt} \tilde{Y}_{g e^{tX}}(f)\Big|_{t=0}$

$= \frac{d^2}{ds\,dt} f(g e^{tX} e^{sY})\Big|_{s=0, t=0}$

$\Rightarrow [X,Y](f) = \frac{d^2}{ds\,dt} f(e^{tX} e^{sY}) - f(e^{sY} e^{tX})\Big|_{s=0, t=0}$

on the other hand

$$(ad_X(Y))(f) = \frac{\partial^2}{\partial s \partial t} f(e^{tX} e^{sY} e^{-tX})\Big|_{t=0,s=0}$$

Baker-Campbell-Hausdorff formula

$$e^{X'} e^{Y'} \sim \exp(X' + Y' + \tfrac{1}{2}[X', Y'] + \text{higher ord term})$$

for $X' = tX$, $Y' = sY$ etc. gives

equality $(ad_X(Y))(f) = [X, Y](f)$.

Prob 5. G conn. & nilpot $\Rightarrow$ $\exp : \mathfrak{g} \to G$ is surj.

Want : img of exp is an open subgroup.

enough to show $\exp(\mathfrak{g})$ is a subgrp

(it always contain a neigh. of e_G.

call it U $\rightsquigarrow$ $g \in \exp(\mathfrak{g})$ $U.g$ would $\subset \exp(\mathfrak{g})$)

By nilpotency $\exists N$ s.t. $[X_1, [X_2, \cdots [X_{N-1}, X_N] \cdots] = 0$

for all $X_1, \cdots, X_N \in \mathfrak{g}$

$\rightsquigarrow$ the BCH formula becomes finite sum.

$$e^X e^Y = \exp(\underbrace{X + Y + \tfrac{1}{2}[X, Y] + \cdots + \text{up to order } N}_{\text{write } P_N(X,Y)})$$

Define a new group structure on $\mathfrak{g}$ by

$X * Y = P_N(X, Y)$. (inverse : $-X$)

so exp is a hom $(\mathfrak{g}, *) \to G$ of top. grps.

$\Rightarrow$ img of exp is a subgrp.

Prob 6 classification of 2-dim Lie algs.

$\mathfrak{g} = \langle X, Y \rangle_{\mathbb{R}}$-span. $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}[X, Y]$.

if $[X, Y] = 0$ $\mathfrak{g}$ is commutative.

if $[X, Y] \neq 0$ we may assume $[X, Y] = Y$

then $\mathfrak{g} \cong \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$ $X \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Y \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$