Invariant bilinear forms

Def. $\mathfrak{g}$: Lie alg, (π, V) : rep of $\mathfrak{g}$

a $\mathfrak{g}$-invariant bilinear form on V is a bilin. form (v, w) on V s.t.

$(\pi_X v, w) + (v, \pi_X w) = 0 \quad v, w \in V$

Why : this corresponds to $(\pi(e^{tX})v, \pi(e^{tX})w)$
$= (v, w)$; differentiate in t.

Examples 1. $\mathfrak{g} = \mathfrak{so}_n = \{X \in M_n(\mathbb{R}) : X^t = -X\}$

$V = \mathbb{R}^n \quad \pi_X v = Xv$ (defining rep. of $\mathfrak{so}_n$)

$\rightsquigarrow$ usual innerprod. is invariant under $\mathfrak{so}_n$

$(Xe_i, e_j) = (\sum X_{ki} e_k, e_j) = X_{ji}$

2. $\mathfrak{g} \subset \mathfrak{gl}_n \quad (X, Y) = \mathrm{Tr}\, XY$ for $X, Y \in \mathfrak{g}$.

$\rightsquigarrow$ this is invar. for the adjoint rep of $\mathfrak{g}$

Key point : $e^{t\,\mathrm{ad}_X}(Y) = e^{tX} Y e^{-tX}$.

both sides are rep of $G < GL_n$,
have the same deriv. $\mathrm{ad}_X(Y) = [X, Y]$

$(e^{t\,\mathrm{ad}_X} Y, e^{t\,\mathrm{ad}_X} Z) = \mathrm{Tr}(e^{tX} YZ e^{-tX})$
$= \mathrm{Tr}\, XY = (Y, Z)$

Or, directly check

$(\mathrm{ad}_X Y, Z) + (Y, \mathrm{ad}_X Z) = \mathrm{Tr}(XYZ - YXZ + YXZ - YZX)$

Def. The Killing form of $\mathfrak{g}$ is $B(X, Y) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}_X \mathrm{ad}_Y)$

(the trace form of $\mathrm{End}(\mathfrak{g})$)

$\rightsquigarrow$ B is invariant for the adj. representation

Thm (C.5 ; Cartan's criterion for solvability)

$\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$, $\mathrm{Tr}(XY) = 0$ for $X, Y \in \mathfrak{g} \Rightarrow \mathfrak{g}$ solvable.

Strategy : $\mathcal{D}(\mathfrak{g})$ nilpotent $\Rightarrow \mathfrak{g}$ solvable

$\underset{\text{nilpot}}{\mathcal{D}(\mathfrak{g})} \to \mathfrak{g} \to \underset{\text{comm.}}{\mathfrak{g}/\mathcal{D}(\mathfrak{g})}$

Step 1. reduction by Engel's thm.

enough to check that each $X \in \mathfrak{g}$ is a nilpotent matrix (0 is the only eigenval.)

Step 2 $\lambda_1, \dots, \lambda_n$ eigenvals of X

$X = X_s + X_n$ semisimple - nilpotent decomp.

X_s : diagonalizable, X_n : nilpot, $X_n X_s = X_s X_n$

(use Jordan normal form $\begin{bmatrix} \lambda & 1 & \\ & \ddots & 1 \\ & & \lambda \end{bmatrix} \rightsquigarrow \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix} + \begin{bmatrix} 0 & 1 & \\ & \ddots & 1 \\ & & 0 \end{bmatrix}$

then $\overline{X_s}$: conjug of X_s $\begin{bmatrix} \bar{\lambda} & & \\ & \ddots & \\ & & \bar{\lambda} \end{bmatrix}$ satisfies

$\mathrm{Tr}\, \overline{X}_s X = \sum_{i=1}^{n} |\lambda_i|^2$

Step 3 $\mathrm{ad}_{\overline{X}_s}$ is a polynomial of ad_X

3-1. ad_{X_s} is a polynom of ad_X

ad_{X_s} is diagonalizable on $\mathfrak{gl}_n(\mathbb{C})$

ad_{X_n} is nilpot.

$\mathrm{ad}_{X_s} \mathrm{ad}_{X_n} = \mathrm{ad}_{X_n} \mathrm{ad}_{X_s}$

$\Rightarrow \mathrm{ad}_{X_s} = (\mathrm{ad}_X)_s$. Y_s is a poly. of Y.

3-2. $\mathrm{ad}_{\overline{X}_s}$ is ...

$\mu_1, \dots, \mu_{n^2}$ eigenvals of ad_{X_s} $(\lambda_i - \lambda_j)$

take a polynom $P(x)$ s.t. $P(\mu_i) = \bar{\mu}_i$

then $\mathrm{ad}_{\overline{X}_s} = \overline{\mathrm{ad}_{X_s}} = P(\mathrm{ad}_{X_s})$

Step 4. $\mathrm{Tr}\, \overline{X}_s X = 0$. ($\underset{\text{Step 1}}{\Rightarrow}$ Claim) for $X \in \mathcal{D}(\mathfrak{g})$

write $X = \sum [Y_i, Z_i]$ $\mathrm{Tr}(\overline{X}_s X) = \sum \mathrm{Tr}(\underbrace{[\overline{X}_s, Y_i]}_{\text{in } \mathfrak{g}} Z_i) = 0$

Th'm (Cartan's criterion for semisimplicity ; C.10)
$\mathfrak{g}$ semisimple $\iff$ the Killing form of $\mathfrak{g}$ is nondegenerate.

Proof Step 1 $\mathfrak{s} = \{ X \in \mathfrak{g} : \forall Y \in \mathfrak{g} \ B(X,Y) = 0 \}$ is an ideal of $\mathfrak{g}$.

$\because$ Take $X \in \mathfrak{s}$, $Y, Z \in \mathfrak{g}$
$\rightsquigarrow$ we want $[Z, X] \in \mathfrak{s}$ i.e. $B([Z,X],Y) = 0$
by invariance $B([Z,X],Y) + \underbrace{B(X,[Z,Y])}_{0} = 0$

Step 2 $\mathfrak{s}$ is solvable
Write $ad_{\mathfrak{s}} = \{ ad_X : X \in \mathfrak{s} \} \subset End(\mathfrak{g})$
- $B(X,Y) = 0$ means $Tr_{\mathfrak{g}}(ad_X \, ad_Y) = 0$.
$\Rightarrow$ $ad_{\mathfrak{s}}$ is solvable by prev Th'm.
- $\mathfrak{s} \to ad_{\mathfrak{s}}$, $X \mapsto ad_X$ has a commutative kernel (equal to $\mathfrak{z}(\mathfrak{g}) = \{ X \in \mathfrak{g} : [X, \forall Y] = 0 \}$

Step 3 "$\Rightarrow$" of Th'm
$\mathfrak{g}$ does not have a solvable ideal $\Rightarrow$ $\mathfrak{s} = 0$.
i.e. $\forall X \neq 0 \ \exists Y \neq 0$ $B(X,Y) \neq 0$.

Step 4 $Rad(\mathfrak{g}) \neq 0 \Rightarrow \exists$ commutative ideal $\mathfrak{h} \triangleleft \mathfrak{g}$
$\because$ generally • $\mathfrak{h} \triangleleft \mathfrak{g} \Rightarrow \mathcal{D}(\mathfrak{h}) \triangleleft \mathfrak{g}$ by Jacobi id.
• $0 \neq \mathfrak{h}$ solvable $\Rightarrow \exists k$ s.t. $\mathcal{D}^k(\mathfrak{h}) \neq 0$ comm.
(take k s.t. $\mathcal{D}^k(\mathfrak{h}) \neq 0$, $\mathcal{D}^{k+1}(\mathfrak{h}) = 0$)
$\Rightarrow \exists k$ $\mathcal{D}^k(Rad(\mathfrak{g})) \neq 0$, comm.

Step 5 "$\Leftarrow$" of Th'm.
By Step 4, enough to have: $B(X,Y)$ nondeg.
$\Rightarrow$ no commutative ideal in $\mathfrak{g}$.

Suppose $\mathfrak{a} \triangleleft \mathfrak{g}$ commutative. $X \in \mathfrak{a}$, $Y \in \mathfrak{g}$
$ad_X \, ad_Y : \mathfrak{g} \to \mathfrak{a} \to 0$

(cont.) so $\mathrm{ad}_X \mathrm{ad}_Y$ has a matrix pres. of the form

$$\begin{array}{cc} \left[\begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array}\right] & \begin{array}{l} \mathfrak{a} \\ \text{compl} \end{array} \\ \begin{array}{cc} \mathfrak{a} & \text{compl} \end{array} & \end{array} \rightsquigarrow \mathrm{Tr}\, \mathrm{ad}_X \mathrm{ad}_Y = 0$$

i.e. $B(X, Y) = 0$

Cor. $\mathfrak{g}$ semisimple $\Leftrightarrow$ $\exists$ simple $\mathfrak{g}_1, \dots, \mathfrak{g}_n$
$\mathfrak{g} \cong \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n$.
(recall $\dim \mathfrak{g}_i > 1$, no ideal)

Proof $\Leftarrow$: any ideal is of the form
$\mathfrak{g}_{i_1} \oplus \dots \oplus \mathfrak{g}_{i_k}$ $\quad 1 \leq i_1 < \dots < i_k \leq n$.

$\Rightarrow$: enough to show: $\mathfrak{h} \triangleleft \mathfrak{g} \Rightarrow \exists\, \mathfrak{h}' \triangleleft \mathfrak{g}$
s.t. $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{h}'$ as Lie algs.
(induction on dimension)

$\mathfrak{h}' = \{ X \in \mathfrak{g} : B(X, Y) = 0 \text{ for } Y \in \mathfrak{h} \}$

- $\mathfrak{h}' \triangleleft \mathfrak{g}$ by invariance of B
- $\mathfrak{h} \cap \mathfrak{h}'$ is solvable by Cartan's criterion

$\Rightarrow$ $\mathfrak{h} \cap \mathfrak{h}' = 0$ by semisimplicity of $\mathfrak{g}$