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Complete reducibility for semisimple Lie algs.

(π, V) : rep of $\mathfrak{g}$.

⇝ can we find irreducible reps of $\mathfrak{g}$ satisfying $(\pi, V) \cong (\pi_1, V_1) \oplus \cdots \oplus (\pi_k, V_k)$?

- No for comm. algs : $\mathfrak{g} = \mathbb{R}X$.

if $V = \mathbb{R}^2$, $\pi_X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, then $W = \{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \}$ is $\mathfrak{g}$-inv. but no decomp $V \cong W \oplus W'$ for some $\mathfrak{g}$-inv W'.

- Yes for semisimple Lie algs.

Ex. irred. decomp. of $(\pi = \mathrm{ad}, V = \mathfrak{g})$ gives decomp $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ $\mathfrak{g}_i$: simple.

Thm (C.15) $\mathfrak{g}$: semisimple complex Lie als (π, V) : rep of $\mathfrak{g}$

$W \subset V$ $\mathfrak{g}$-inv. subsp.

Then $\exists$ $\mathfrak{g}$-inv subsp. $W' \subset V$ s.t. $V = W \oplus W'$.

Proof. Step 0 we may assume $\mathfrak{g} \subset \mathfrak{gl}(V)$

$\because$ kernel of π is an ideal

$\Rightarrow \ker \pi = \mathfrak{g}_{i_1} \oplus \cdots \oplus \mathfrak{g}_{i_m}$ $1 \le i_1 < \cdots < i_m \le k$.

so $\{ \pi_X : X \in \mathfrak{g} \} \cong \bigoplus_{j \notin \{i_1, \ldots, i_m\}} \mathfrak{g}_j$

Step 1. Put $B_V(X, Y) = \mathrm{Tr}_V(XY)$ for $X, Y \in \mathfrak{g}$

Then B_V is nondeg. on $\mathfrak{g}$

$\because$ $\mathfrak{f} = \{ X \in \mathfrak{g} : \forall Y \ B_V(X, Y) = 0 \} \triangleleft \mathfrak{g}$ by invariance of B_V

Cartan's criterion $\Rightarrow$ $\mathfrak{f}$ is solvable

$\underset{\mathfrak{g} \text{ is s.s.}}{\Longrightarrow}$ $\mathfrak{f} = 0$

Step 2 Put $C_V = \sum_{i=1}^{d} X_i X^i$ with $(X_i)_{i=1}^{d}$ basis of $\mathfrak{g}$, $(X^i)_i$ dual basis for B_V.

Then C_V is an intertwiner for $\mathfrak{g}$

(cont) we want $[Y, C_V] = 0$ for $Y \in \mathfrak{g}$

$$[Y, C_V] = \sum_i [Y, X_i]X^i + X_i[Y, X^i]$$

$$= \sum_{i,j} B_V([Y, X_i], X^j)\, X_j X^i + B_V(X_j, [Y, X^i])X_i X^j$$

by invariance of B_V second term is

$-B_V([Y, X_j], X^i)\, X_i X^j$ w/ cancel with first term

Step 3 claim when $\dim W = \dim V - 1$, W irred

we show $W' = \ker C_V$ would work

- $C_V|_W$ is a scalar by Schur's lemma and irreducibility of W
- $\mathfrak{g} \overset{\bar\pi}{\curvearrowright} V/W$ should be trivial: V/W is 1-dim, so $\{\bar\pi_X : X \in \mathfrak{g}\}$ is comm.

  i.e. $\bar\pi_X = 0$ for $X \in \mathcal{D}(\mathfrak{g})$ but $\mathfrak{g} = \mathcal{D}(\mathfrak{g})$

  $\Rightarrow C_V$ induces 0 map on V/W i.e. $\operatorname{Im} C_V \subset W$
- $C_V|_W = \frac{\dim \mathfrak{g}}{\dim W} \mathrm{Id}$: $\mathrm{Tr}_V(C_V) = \sum B_V(X_i, X^i) = \dim \mathfrak{g}$

So $\ker C_V$ is 1-dim, complement of W & $\mathfrak{g}$-inv.

We do the rest by induction on $\dim V$

Step 4 claim when $\dim W = \dim V - 1$.

If W is not irred, take $Z \subset W$ nonzero inv.

V/Z has a rep of $\mathfrak{g}$, $W/Z \subset V/Z$ codim 1

$\Rightarrow V/Z \simeq W/Z \oplus Y_0$ for some $Y_0 \subset V/Z$

Put Y = inv. img of Y_0 in V

so $Z \subset Y$ $\mathfrak{g}$-inv. $\dim Y < \dim V$

$\underset{\text{ind. hyp.}}{\Rightarrow} \exists\, Y_1 \subset Y$ $\mathfrak{g}$-inv. $Y = Z \oplus Y_1$

Y_1 is a compl. of W

Step 5 claim when W irred.

We show $\exists\, T : V \to W$ $\mathfrak{g}$-intertwiner, $T|_W = \mathrm{Id}$.

Then $V = W \oplus \ker T$.

- By Schur's lem. $\mathrm{End}_{\mathfrak{g}}(W)$ is 1-dim.
- $\mathrm{Hom}(V, W) \xrightarrow{\text{res}} \mathrm{End}(W)$ (w/o $\mathfrak{g}$-inv.) is surjective.

$\rightsquigarrow$ put U = inv. img of $\mathrm{End}_{\mathfrak{g}}(W)$ in $\mathrm{Hom}(V, W)$

$U \xrightarrow{\text{res.}} \mathrm{End}_{\mathfrak{g}}(W)$ has codim 1 kernel U_1.

U_1 is $\mathfrak{g}$-invar. (for $\mathrm{ad}_X T = XT - TX$)

$\underset{\text{Step 4}}{\Rightarrow}$ $U = U_1 \oplus U_2$ for $\mathfrak{g}$-inv. $U_2 < U$

$\dim U_2 = 1$ so $\mathrm{ad}\, X|_{U_2} = 0$ i.e. $U_2 \subset \mathrm{Hom}_{\mathfrak{g}}(V, W)$

Take $T \in U_2$ s.t. $\mathrm{res}\, T = \mathrm{Id}_W$.

Step 6 claim for arbitrary W

same as Step 3 $\Rightarrow$ Step 4. $\blacksquare$

Summary: reps of a semisimple Lie alg are determined by irred. subreps. enough to know the list of irred reps.

Ex. $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$: (π^n, V_n) $\dim V_n = n+1$
$n = 0, 1, \ldots$

Semisimple-nilpotent decomp in semisimple Lie algs.

Prop (C.17) $\mathfrak{g} \subset \mathfrak{gl}(V)$ complex semisimple $X \in \mathfrak{g}$.

$X = X_s + X_n$ semisimple-nilpot decomp.

Then $X_s, X_n \in \mathfrak{g}$.

Rem. this does not hold for other class

$X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $\mathfrak{g} = \mathbb{R}X \subset \mathfrak{gl}_2(\mathbb{R})$. $X_s = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $X_n = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Proof Step 1 $\tilde{\mathfrak{g}} = \{ Y \in \mathfrak{gl}(V) : [Y, \mathfrak{g}] \subset \mathfrak{g} \}$

$\mathfrak{s}_W = \{ Y \in \mathfrak{gl}(V) : YW \subset W,\ \mathrm{Tr}_W(Y) = 0 \}$ for $W \subset V$

Then $\mathfrak{g} = \tilde{\mathfrak{g}} \cap \left(\bigcap_{\substack{W \subset V \\ \mathfrak{g}\text{-inv.}}} \mathfrak{s}_W \right)$.

$\because$ $\subset$ follows from $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$, $\mathfrak{g}|_W \subset \mathfrak{sl}(W)$ etc.

$\supset$: put $\mathfrak{g}' = \tilde{\mathfrak{g}} \cap \left(\bigcap_{W: \mathfrak{g}\text{-inv}} \mathfrak{s}_W \right)$

by complete reducibility $\mathfrak{g}' = \mathfrak{g} \oplus U$ as

rep. of $\mathfrak{g}$. we want to show $U = 0$

Enough to show: $Y \in U$, $W \subset V$ irred $\mathfrak{g}$-mod.

$\Rightarrow Y|_W = 0$ ($YW \subset W$ by $Y \in \mathfrak{s}_W$)

$[Y, \mathfrak{g}] \subset \mathfrak{g}$ from $Y \in \tilde{\mathfrak{g}}$

$[\mathfrak{g}, Y] \subset U$ from $\mathfrak{g}$-invariance of U

$\Rightarrow [Y, \mathfrak{g}] \in \mathfrak{g} \cap U = 0$ i.e. Y is a $\mathfrak{g}$-intertw.

$\Rightarrow Y|_W$ is scalar by Schur's lem.

$\mathrm{Tr}_W Y = 0$ implies $Y = 0$.

We show $X_n \in \tilde{\mathfrak{g}}$, $X_n \in \mathfrak{s}_W$ separately.

($\Rightarrow X_n \in \mathfrak{g}$, $X_s = X - X_n$ also in $\mathfrak{g}$)

Step 2 $X_n \in \mathfrak{s}_W$ for $\mathfrak{g}$-inv. $W \subset V$

X_n is a polynom. of $X \Rightarrow X_n W \subset W$.

X_n nilpotent $\Rightarrow \mathrm{Tr}_W X_n = 0$

Step 3 $X_n \in \tilde{\mathfrak{g}}$

$\mathrm{ad}_{X_n} = (\mathrm{ad}_X)_n$ so ad_{X_n} is a polynom of ad_X

$\Rightarrow \mathrm{ad}_{X_n}(\mathfrak{g}) \subset \mathfrak{g}$ $\square$