§3 Classification of complex simple Lie algebras

Main ref. Fulton–Harris § 14, 21 for general theory
§§ 11 – 20 for concrete examples.

Goal: understand cplx simple Lie algs

Motivating example: $\mathfrak{sl}_n\mathbb{C} = \{X \in M_n(\mathbb{C}) : \operatorname{Tr} X = 0\}$

basis of $\mathfrak{sl}_n(\mathbb{C})$:

- diagonal part $H_i = E_{i,i} - E_{i+1,i+1}$ $i = 1, \dots, n-1$
- off-diag part $E_{i,j}$ $i \neq j$ 1 at (i,j)-th comp.

bracket rel: $[H_i, H_j] = 0$, $[H_i, E_{j,k}] = (\delta_{i,j} - \delta_{i+1,j} - \delta_{k,i} - \delta_{i,i+1})E_{j,k}$

$[E_{i,j}, E_{k,l}] = \delta_{j,k} E_{i,l} - \delta_{l,i} E_{k,j}$

$E_{i,i} - E_{k,k} = H_i + H_{i+1} + \dots + H_{k-1}$ if $i = l < j = k$

obs: • diag. part span a commutative alg. $\mathfrak{h} = \langle H_i \rangle_{i=1}^{n-1}$

• $E_{i,j}$ is a joint eigenvector for $H \in \mathfrak{h}$
with different eigenvals for diff. (i,j)

• bracket of eigenvectors ⇝ another eigenvec.
(or 0, or in $\mathfrak{h}$).

smallest system of eigenvectors.

$E_i = E_{i,i+1}$, $F_i = E_{i+1,i}$ $(i = 1, \dots, n-1)$

these will generate $\mathfrak{sl}_n(\mathbb{C})$ e.g. $[E_1, E_2] = E_{1,3}$, …

⇝ it's enough to understand how to produce
other joint eigenvals of $\mathfrak{h} \overset{\mathrm{ad}}{\curvearrowright} \mathfrak{sl}_n\mathbb{C}$
and how they interact under bracket of corresp
eigenvectors.

Def. a <u>Cartan subalgebra</u> of $\mathfrak{g}$ is $\mathfrak{h} < \mathfrak{g}$ s.t.

- commutative
- if $Y \in \mathfrak{g}$ comms w/ $\mathfrak{h}$ then $Y \in \mathfrak{h}$.
- $\forall X \in \mathfrak{h}$ $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable

Prop. $\mathfrak{g}$ (complex semi) simple Lie alg.

(π, V) fin. dim rep. of $\mathfrak{g}$.

$X \in \mathfrak{h}$

Then π_X is diagonalizable.

Proof. $\exists Y$ s.t. $(\pi_X)_s = \pi_Y$ (semisimple - nilpot. decomp.)

ad_{π_Y} (on $\pi(\mathfrak{g})$) is the semisimple part of

ad_{π_X} (on $\pi(\mathfrak{g})$) but ad_{π_X} is diagonalizable

$\Rightarrow$ $\mathrm{ad}_{\pi_Y} = \mathrm{ad}_{\pi_X}$ $\Rightarrow$ $X - Y$ is in the center $= 0$

we may assume π is irred, and faithful.

Cor. V has a basis of joint eigenvectors

$v_1, \dots, v_m$ s.t. $\pi_X v_i = \lambda_i(X) v_i$ $(X \in \mathfrak{h})$

Rem. $\lambda_i(X)$ is a linear form on $\mathfrak{h}$.

Def. <u>weights</u> of (π, V) : $\lambda_1, \dots, \lambda_m$ as above.

weights of $(\mathfrak{g}, \mathfrak{h})$: $\lambda \in \mathfrak{h}^*$ that appear as above.

Rem. weights form a subgroup of $\mathfrak{h}^*$:

$\lambda_1, \dots, \lambda_m$ weights of (π, V)

$\lambda'_1, \dots, \lambda'_n$ ~ (π', V')

$\Rightarrow$ $\lambda_i + \lambda'_j$ weights of $(\pi \otimes \pi', V \otimes V')$

with $(\pi \otimes \pi')_X = \pi_X \otimes \mathrm{id} + \mathrm{id} \otimes \pi'_X$.

$-\lambda_i$ weights of (π^c, V^*), $\pi^c_X = -\pi_X^t$

Fact weights form a <u>lattice</u> of $\mathfrak{h}^*$ $(\simeq \mathbb{Z}^{\dim \mathfrak{h}})$

Def. <u>roots</u> of $(\mathfrak{g}, \mathfrak{h})$: nonzero weights for the adjoint rep $(\pi = \mathrm{ad}, V = \mathfrak{g})$

So $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha : \mathrm{root}} \mathfrak{g}_\alpha \right)$, $\mathfrak{g}_\alpha = \{ Y : \forall X \in \mathfrak{h}\ [X, Y] = \alpha(X) Y \}$

Example. $\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}H \oplus \mathbb{C}E \oplus \mathbb{C}F$
$\qquad\qquad\qquad\quad \mathfrak{h} \qquad \mathfrak{g}_\alpha \qquad \mathfrak{g}_{-\alpha}$

$\alpha(H) = 2$ from $[H, E] = 2E$, $[H, F] = -2F$.

fin. dim irred. reps (π^n, V_n) $V_n = \langle x^n, x^{n-1}y, \dots, y^n \rangle$

$\rightsquigarrow$ weights $\lambda_n, \lambda_{n-2}, \dots, \lambda_{-n}$; $\lambda_k(H) = k$.

We want to get a geometric / combinatorical structure from roots

Fact 1 α : root $\Rightarrow$ $\dim \mathfrak{g}_\alpha = 1$ i.e. joint eigen vector for ad_x $(x \in \mathfrak{h})$ is unique for each root.

Fact 2 α : root $\Rightarrow$ $\mathfrak{s}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a 3-dim subalg of $\mathfrak{g}$, $\mathfrak{s}_\alpha \simeq \mathfrak{sl}_2 \mathbb{C}$.

Ex. $\mathfrak{g}_\alpha = \mathbb{C}E_{j,k}$ in $\mathfrak{g} = \mathfrak{sl}_n \mathbb{C}$

$\mathfrak{s}_\alpha = \mathbb{C}(E_{j,j} - E_{k,k}) \oplus \mathbb{C}E_{j,k} \oplus \mathbb{C}E_{k,j}$

Notation $H_\alpha, E_\alpha, F_\alpha \in \mathfrak{s}_\alpha$ corr. to $H, E, F \in \mathfrak{sl}_2\mathbb{C}$