Root system from simple Lie algebras

Recall

$\mathfrak{g}$: complex simple Lie alg.

$\mathfrak{h}$: Cartan subalg. of $\mathfrak{g}$.

$\rightsquigarrow$ roots : $\alpha \in \mathfrak{h}^*$ s.t. $\exists\, 0 \neq Y \in \mathfrak{g}$
$\forall X \in \mathfrak{h}$ $[X, Y] = \alpha(X) Y$

We want to find a Euclidean space E
(real vec. sp. with pos. def. inn. prod.)
that contains the roots of $(\mathfrak{g}, \mathfrak{h})$
$\hookrightarrow$ put $R := \{\alpha : \text{roots}\}$

Intermediate steps

1. find a copy of $\mathfrak{sl}_2\mathbb{C}$ in $\mathfrak{g}$ for each $\alpha \in R$
   in fact $\underset{\substack{\Psi \\ H_\alpha}}{[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \simeq \mathfrak{sl}_2\mathbb{C}$.

2. $E \subset \mathfrak{h}^*$ real dual of $\mathfrak{h}_0 = \mathbb{R}\text{-span}\langle H_\alpha : \alpha \in R \rangle$.

3. the Killing form $B(X, Y) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}_X\, \mathrm{ad}_Y)$
   is pos. def. on $\mathfrak{h}_0$ $\rightsquigarrow$ Euclidean str. on E.

Th'm (§ D.1) $\alpha \in R \Rightarrow [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \simeq \mathfrak{sl}_2(\mathbb{C})$.

Proof Step 1. $-\alpha \in R$.

look at $B(X, Y)$ $0 \neq X \in \mathfrak{g}_\alpha$ this is nonzero
for some $Y \in \mathfrak{g}$.

$Y \in \mathfrak{g}_\beta$, $\beta \neq -\alpha \Rightarrow B(X, Y) = 0$ same w/ $Y \in \mathfrak{h}$.

$\because$ by invariance $B(\underset{\alpha(Z)X}{[Z, X]}, Y) = -B(X, \underset{\beta(Z)Y}{[Z, Y]})$

so $B(X, Y) \neq 0$ for some $Y \in \mathfrak{g}_{-\alpha}$.

Step 2 $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_{-\alpha} \Rightarrow [X, Y] = B(X, Y) T_\alpha$, $B(T_\alpha, Z) = \alpha(Z)$ $Z \in \mathfrak{h}$

look at $B([X, Y], Z)$ for $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_{-\alpha}, Z \in \mathfrak{h}$

this is $-B(Y, [X, Z]) = \alpha(Z) B(Y, X)$

Step 3 $\alpha(T_\alpha) \neq 0$

Take $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ s.t. $B(X,Y) \neq 0$

Put $\tilde{\mathfrak{s}}_\alpha = \langle X, Y, T_\alpha \rangle$. this is a Lie subalg

$[X,Y] = B(X,Y)T_\alpha$, $[T_\alpha, X] = \alpha(T_\alpha)X$, $[T_\alpha, Y] = -\alpha(T_\alpha)Y$

If $\alpha(T_\alpha) = 0$, $\mathcal{D}(\tilde{\mathfrak{s}}_\alpha) = \langle T_\alpha \rangle = \mathfrak{z}(\tilde{\mathfrak{s}}_\alpha)$ so

$\tilde{\mathfrak{s}}_\alpha$ is nilpotent

Lie's th'm $\Rightarrow$ $\mathfrak{g}'$ solvable, (π, V) rep. of $\mathfrak{g}'$

$X \in \mathcal{D}(\mathfrak{g}) \Rightarrow \pi_X$ is nilpot.

So ad_{T_α} would be nilpotent.

But we also know ad_{T_α} diagonalizable, nonzero.

Step 4 $\mathfrak{sl}_2\mathbb{C} \to [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ hom.

choose $E' \in \mathfrak{g}_\alpha$, $F' \in \mathfrak{g}_{-\alpha}$ s.t. $\alpha([E', F']) = 2$

Then $\mathfrak{s}_\alpha = \langle E', F', H' = [E', F'] \rangle \simeq \mathfrak{sl}_2(\mathbb{C})$

Step 5 $\dim \mathfrak{g}_\alpha = 1$

Put $V = \mathfrak{h} \oplus \Big(\bigoplus_{\substack{k \in \mathbb{C} \\ k\alpha \in R}} \mathfrak{g}_{k\alpha} \Big)$. we want $V = \mathfrak{h} \oplus \mathfrak{s}_\alpha$

V is stable under ad_X ($X \in \mathfrak{s}_\alpha$)

$\rightsquigarrow$ decomp. as direct sum of copies

(π_n, V_n) $(n+1)$-dim irred. rep of $\mathfrak{sl}_2(\mathbb{C})$

0-weigh space for H' is $\mathfrak{h}$ by Step 3

triv-rep of $\mathfrak{s}_\alpha$ given by $\ker \alpha \subset \mathfrak{h}$
codim 1.

$\rightsquigarrow$ there can be only one copy of V_1 ($\simeq_{\mathrm{ad}} \mathfrak{s}_\alpha$) $\square$

Rem. $\alpha \in R \Rightarrow \underset{k \neq \pm 1}{k\alpha \notin R}$ from Step 5

Ex. $\mathfrak{g} = \mathfrak{sl}_n$ α : root for $E_{i,j}$ $(i \neq j)$

$\Rightarrow$ $H_\alpha = E_{i,i} - E_{j,j}$, $E_\alpha = E_{i,j}$, $E_{-\alpha} = E_{j,i}$

span a copy of $\mathfrak{sl}_2$

Notn. $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha = E_{-\alpha} \in \mathfrak{g}_{-\alpha}$

corresp. to $H, E, F \in \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{s}_\alpha = \langle H_\alpha, E_\alpha, F_\alpha \rangle$

Rem (π, V) rep of $\mathfrak{g}$, $\lambda \in \mathfrak{h}^*$ weight for (π, V). (so $\exists v \neq 0 \in V$ $\forall X \in \mathfrak{h}$ $\pi_X v = \lambda(X) v$)

$\Rightarrow$ $\pi|_{\mathfrak{s}_\alpha}$ is a rep of $\mathfrak{sl}_2 \mathbb{C}$

$\Rightarrow$ $\lambda|_{\mathbb{C} H_\alpha}$ is a weight of $(\mathfrak{sl}_2 \mathbb{C}, \mathbb{C} H)$

$\Rightarrow$ $\lambda(H_\alpha) \in \mathbb{Z}$. (from weights of (π_n, V_n))

Def. <u>weight lattice</u> of $(\mathfrak{g}, \mathfrak{h})$

$\Lambda_W = \{ \lambda \in \mathfrak{h}^* : \lambda(H_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in R \}$.

<u>root lattice</u> Λ_R = span of R in Λ_W.

$E = \mathbb{R} \Lambda_W \subset \mathfrak{h}^*$, $\mathfrak{h}_0 = \mathbb{R}\text{-span}\langle H_\alpha : \alpha \in R \rangle \subset \mathfrak{h}$.

Rem. E is the (real) dual space of $\mathfrak{h}_0$.

Prop. 1. $B(X, Y) \in \mathbb{R}$ for $X, Y \in \mathfrak{h}_0$

2. $B(X, X) \geq 0$, $= 0$ iff $X = 0$ for $X \in \mathfrak{h}_0$

Proof 1. Enough to have $B(H_\alpha, H_\beta) \in \mathbb{R}$ for $\alpha, \beta \in R$

ad_{H_α} and ad_{H_β} are simultaniously diagonalizable

- eigenval 0 on $\mathfrak{h}$, $\gamma(H_\alpha)$ on $\mathfrak{g}_\gamma$ $(\gamma \in R)$

$\gamma(H_\alpha) \in \mathbb{Z}$ by classification of $\mathfrak{sl}_2(\mathbb{C})$ reps

$\mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}_{H_\alpha} \mathrm{ad}_{H_\beta}) = \sum_{\gamma \in R} \gamma(H_\alpha)\gamma(H_\beta) \in \mathbb{Z}$

so we actually get $B(H_\alpha, H_\beta) \in \mathbb{Z}$.

2. ad_X for $X \in \mathfrak{h}_0$ acts on $\mathfrak{g}_\gamma$ by real number mult. $(\gamma(X) \in \mathbb{R})$

$B(X, X) = \sum_{\gamma \in R} \gamma(X)^2 \geq 0$, $= 0$ holds iff

$\mathrm{ad}_X = 0$ i.e. $X \in \mathfrak{z}(\mathfrak{g}) = 0$ $\boxed{\scriptstyle{}}$

Cor. by duality we get a pos. def. inn. prod. on E

Example $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ $\quad$ $\mathfrak{h}$ = diag matrices in $\mathfrak{sl}_3(\mathbb{C})$

$= \langle H_1 = E_{1,1} - E_{2,2},\ H_2 = E_{2,2} - E_{3,3} \rangle$

ad_{H_1} : $E_{1,2} \mapsto 2E_{1,2}$, $E_{2,1} \mapsto -2E_{2,1}$

$E_{2,3} \mapsto -E_{2,3}$, $E_{3,2} \mapsto E_{3,2}$

$E_{1,3} \mapsto E_{1,3}$, $E_{3,1} \mapsto -E_{3,1}$

ad_{H_2} : $E_{1,2} \mapsto -E_{1,2}$, $E_{2,1} \mapsto E_{2,1}$

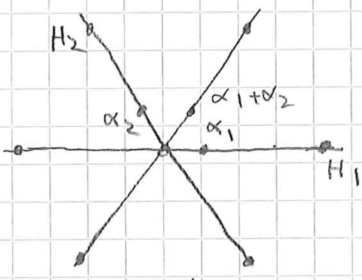
$E_{2,3} \mapsto 2E_{2,3}$, $E_{3,2} \mapsto -2E_{3,2}$

$E_{1,3} \mapsto E_{1,3}$, $E_{3,1} \mapsto -E_{3,1}$

$B(H_1, H_1) = 2 \times 4 + 4 \times 1 = 12 = B(H_2, H_2)$

$B(H_1, H_2) = 4 \times (-2) + 2 \times 1 = -6$

α_1 : root for $E_{1,2}$, α_2 : root for $E_{2,3}$



$\alpha_1(H_1) = 2$, $\alpha_1(H_2) = -1$

$\alpha_2(H_1) = -1$, $\alpha_2(H_2) = 2$

$\alpha_1 + \alpha_2$: root for $E_{1,3}$