Recap. from cplx simple Lie alg $\mathfrak{g}$,
Cartan subalg $\mathfrak{h} < \mathfrak{g}$

- roots $\alpha \in R$ eigenval functional
  $Y \in \mathfrak{g}_\alpha \subset \mathfrak{g}$: $[X, Y] = \alpha(X) Y$ $X \in \mathfrak{h}$
- copy of $\mathfrak{sl}_2\mathbb{C}$: $\langle H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], E_\alpha \in \mathfrak{g}_\alpha, F_\alpha \in \mathfrak{g}_{-\alpha} \rangle$
- real subsp. $\mathfrak{h}_0 = \langle H_\alpha : \alpha \in R \rangle_\mathbb{R} \subset \mathfrak{h}$
- Killing form B : pos. def. on $\mathfrak{h}_0$
- weight lattice $\Lambda_W = \{ \beta \in \mathfrak{h}^* : \forall \alpha \in R \ \beta(H_\alpha) \in \mathbb{Z} \}$
  $R \subset \Lambda_W \subset E = \mathbb{R}$-span of Λ_W. $\in$ Euclidean sp. by inverse of B.

We know : $\alpha \in R \Rightarrow k\alpha \in R$ iff $k = \pm 1$.

More symmetry :

$$s_\alpha(v) = v - \frac{2(\alpha, v)}{(\alpha, \alpha)} \alpha \quad \text{for } v \in E$$

reflection along the hyperplane $\alpha^\perp = \{ m \in E : (\alpha, m) = 0 \}$

Def. the Weyl group of $(\mathfrak{g}; \mathfrak{h})$ is the subgroup of $O(E)$ generated by s_α $(\alpha \in R)$

Prop 1. $\alpha \in R$: $s_\alpha(v) = v - v(H_\alpha)\alpha$ for $v \in E$.

i.e. $v(H_\alpha) = \dfrac{2(\alpha, v)}{(v, v)}$

Proof Step 1 $\alpha^\perp = \{ m \in E : m(H_\alpha) = 0 \}$

Recall $\exists T_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ s.t. $B(T_\alpha, Z) = \alpha(Z)$ $(Z \in \mathfrak{h})$

the isom $E \cong \mathfrak{h}_0$ given by B maps α to T_α

$(m, \alpha) = 0 \Leftrightarrow Z$ corr. to m satisfies $B(Z, T_\alpha) = 0$

$\Leftrightarrow m(T_\alpha) = 0 \Leftrightarrow m(H_\alpha) = 0$

Step 2 $s_\alpha(v) = v - \dfrac{2 v(H_\alpha)}{\alpha(H_\alpha)} \alpha$

Step 3 $\alpha(H_\alpha) = 2$ □

Prop 2 (π, V) rep. of $\mathfrak{g}$; put

$X = \mathrm{supp}(\pi, V) = \{ \lambda \in \mathfrak{h}^* \text{ weight of } V \}$.

Then $s_\alpha X = X$

Proof S.Q. Fix $\lambda \in X$, take max $p, q \in \mathbb{N}$ s.t.

$I = \{ \lambda + p\alpha, \lambda + (p-1)\alpha, \dots, \lambda - q\alpha \} \subset X$

Step 1 $(\lambda + p\alpha)(H_\alpha) = -(\lambda - q\alpha)(H_\alpha)$

From classification of irred reps of $\mathfrak{sl}_2(\mathbb{C})$

Step 2 $\lambda(H_\alpha) = q - p$.

Step 3 $s_\alpha(\lambda) \in X$

$s_\alpha(\lambda) = \lambda + (p-q)\alpha \in I$. ⊠

Cor $R \subset E$ is invariant under the Weyl group

Def. a <u>root system</u> is given by

- E : Euclidean space
- $R \subset E$: finite subset ; such that

1. R spans E
2. $\alpha \in R$, $k \in \mathbb{R}$: $k\alpha \in R \Leftrightarrow k = \pm 1$
3. $s_\alpha(v) = v - \frac{2(\alpha, v)}{(v, v)} \alpha$ $(\alpha \in R)$ maps R to R
4. $n_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for $\alpha, \beta \in R$

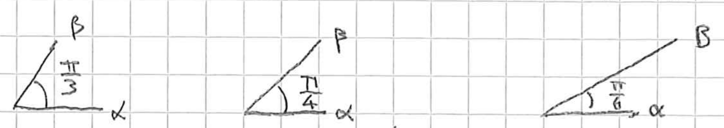
Prop 3 $\alpha, \beta \in R \Rightarrow$ the angle between α and β is $0, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$, or $\pi -$ (one of the left)

Proof $n_{\alpha,\beta}\, n_{\beta,\alpha} = \frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = 4\cos^2\theta$

for the angle θ between α and β

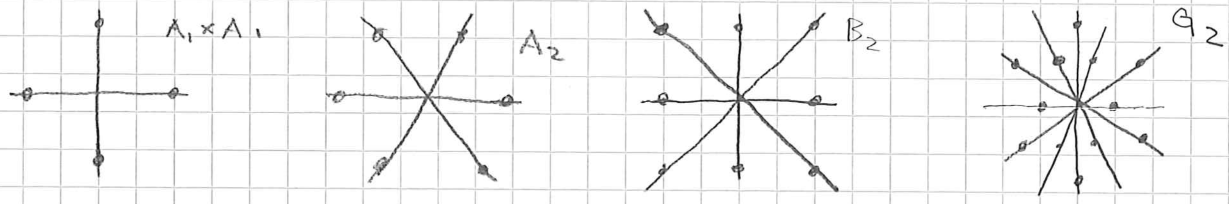
$\Rightarrow \cos\theta = \pm \frac{\sqrt{n}}{4}$ for $n \in \mathbb{N}$; the only possible values are $n = 0, 1, 2, 3, 4$ corresponding to those in the claim ⊠

Rem we also have restrictions on the length



$\|\alpha\| = \|\beta\|$ $\|\beta\| = \sqrt{2}^{\pm 1} \|\alpha\|$ $\|\beta\| = \sqrt{3}^{\pm 1} \|\alpha\|$

Ex. If $\dim E = 2$, R is on of :



Rem $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\mathfrak{h} = \text{diag}$. gives the A_2 system (previous time)

$\mathfrak{g} = \mathfrak{so}_5(\mathbb{C})$ or $\mathfrak{spin}_4(\mathbb{C})$ gives the B_2 system "exercise" ||?

$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{h} = \mathbb{C}H \oplus \mathbb{C}H$ gives the $A_1 \times A_1$ system

Rem. (E_1, R_1), (E_2, R_2) root systems

$\Rightarrow (E_1 \oplus E_2, R_1 \times \{0\} \cup \{0\} \times R_2)$ is again a root sys.
(given by $E_1 \times E_2$)

Def. (E, R) is an <u>irreducible</u> root system if there is no nontriv decomp $(E, R) \cong (E_1, R_1) \times (E_2, R_2)$

Goal : $\{$cplx simple Lie alg s$\}$ / isom $\overset{1:1}{\longleftrightarrow}$ $\{$irred. root sys$\}$ / isom

Todo

- define (simple) Lie alg s from (irred) root sys.
- check the independence of root sys. on the choice of Cartan subalg s. $\mathfrak{h} < \mathfrak{g}$
- $(\mathfrak{g}, \mathfrak{h}) \rightsquigarrow$ root sys., root sys $\rightsquigarrow (\mathfrak{g}, \mathfrak{h})$ are inverse to each other