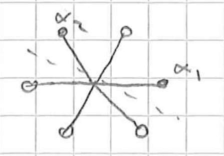
Classification of irred. root systems

Overview : (E, R) (irr.) root sys

$\rightsquigarrow$ 1. choose a subset $\Pi \subset R$ that becomes a basis of E.

$\Pi \subset$ half-space of E, as wide as possible



2. draw a graph (Dynkin diagram) expressing the matrix $\left(n_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\right)_{\alpha,\beta}$ Cartan mat

o—o A_2 o⇒o B_2 o⇛o G_2 ...

3. classify the Dynkin diagrams.

o—o— - - —o A_n o—o— · · · —o⇒o B_n o— - - - —o< D_n

$\rightsquigarrow$ classification of irred. root systems.

Def. an order on roots R is given by a decomp $R = R_+ \amalg R_-$ s.t. $\exists$ lin func. $l : E \to \mathbb{R}$

$$R_+ = \{\alpha : l(\alpha) > 0\}, \quad R_- = \{\alpha : l(\alpha) < 0\}$$

$\alpha \in R_+$ is simple if $\nexists\ \beta, \gamma \in R_+$ $\alpha = \beta + \gamma$

Notn. $\Pi = \{\text{simple}\}$

Example root sys of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{h} = \text{diag}$

$R_+ = \{\alpha_{i,j} : \text{root for } E_{i,j}, \ i < j\}$

$R_- = \{\alpha_{i,j}$ ——— $i > j\}$

take $Z = \text{diag}(d_1, \dots, d_n)$ $d_1 > d_2 > \dots > d_n$, $\sum_{i=1}^{n} d_i = 0$

so $\alpha_{i,j}(Z) = d_i - d_j$ > 0 for $\alpha_{i,j} \in R_+$

< 0 for $\alpha_{i,j} \in R_-$

$\Pi = \{\alpha_i = \alpha_{i,i+1} : i = 1, \dots, n-1\}$

Up to normalization $(\alpha_i, \alpha_i) = 2$, $(\alpha_i, \alpha_j) = -1$ $|i-j|$

Recall (Prop I & proof of Prop. 2, Oct 26).

(π, V) rep of $\mathfrak{g}$, λ wght of V, $\alpha \in R$

p, q max int. s.t. $\lambda + p\alpha, \lambda + (p-1)\alpha, \cdots, \lambda - q\alpha$ are weights of V

then $\lambda(H_\alpha) = q - p = \dfrac{2(\lambda, \alpha)}{(\alpha, \alpha)}$

Lem $\alpha, \beta \in R$ $\beta \neq \pm\alpha$ then with $\lambda = \beta$ above

p, q satisfy $p + q \leq 3$, $q - p = n_{\alpha,\beta}$

Proof $q - p = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} = n_{\alpha,\beta}$

Take $\lambda' = \beta - q\alpha$, compute $p' = p + q$, $q' = 0$

$p + q = -(q' - p') = -n_{\alpha,\lambda'}$: can be $0, 1, 2, 3$

depending on the angle between α and λ'.

Prop $\alpha, \beta \in R$.

1) $(\alpha, \beta) > 0 \Rightarrow \alpha - \beta \in R$

2) $(\alpha, \beta) < 0 \Rightarrow \alpha + \beta \in R$

Proof of 1) $s_\alpha(\beta) = \beta - \dfrac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha = \beta - n_{\alpha,\beta}\alpha$

$\beta - \alpha$ sits between β and $s_\alpha(\beta)$

2) follows from 1) by taking $\beta' = -\beta \in R$

Cor $\alpha, \beta \in \Pi = \{\text{simple pos. root}\}$. $\Rightarrow n_{\alpha,\beta} \in \{0, -1, -2, -3\}$

$\because$ $n_{\alpha,\beta} > 0 \Rightarrow$ either $\alpha - \beta$ or $\beta - \alpha$ is positive

$\Rightarrow$ nontriv decomp $\alpha = \beta + (\alpha - \beta)$, $\beta = \alpha + (\beta - \alpha)$

Fact any two orders on R are conj. by the action of the Weyl group

$R = R'_+ \amalg R'_-$ $\Rightarrow \exists\, w \in W$ $R'_+ = wR_+$, $R'_- = wR_-$

So $\Pi' = w\Pi$.

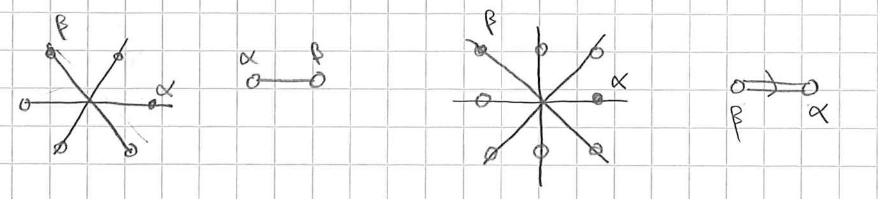
Def. the <u>Dynkin diagram</u> of (E, R) is given by

- vertices : simple positive roots Π
- edges between $\alpha, \beta \in \Pi$: k edges if

$$\frac{(\alpha, \beta)}{\|\alpha\| \cdot \|\beta\|} = -\frac{\sqrt{k}}{2} \qquad (k = n_{\alpha,\beta} n_{\beta,\alpha} \in \{0, 1, 2, 3\})$$

for $k = 2, 3$: put orientation $\overset{\alpha}{\circ} \Rightarrow \overset{\beta}{\circ}$ etc. $\|\alpha\| > \|\beta\|$

Ex.

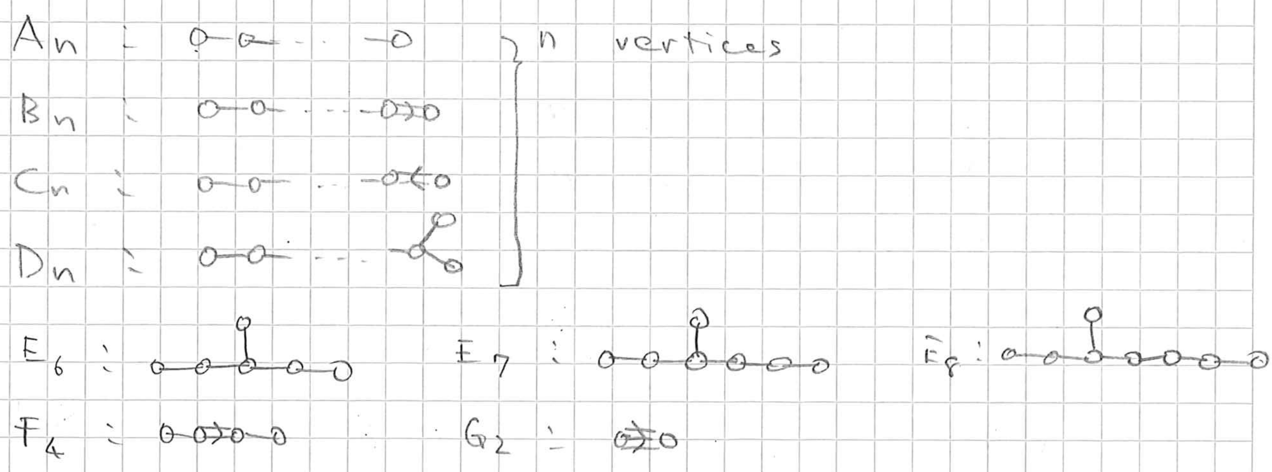


Rem. - Dynkin diag can recover (E, R).

- D-diag for $(E_1, R_1) \times (E_2, R_2)$ = disj. union of D-diags for (E_i, R_i)

Thm $\exists$ irred root systems $\equiv$ conn. D-diagrams

The list of Dynkin diagrams :



Getting restrictions on the shapes

Γ : Dynkin diagram of (E, R), $\Pi \subset R_+$ ...

Prop 2 no loop in Γ

Proof This is equivalent to :

$\forall J \subset \Pi$: at most $2(|J| - 1)$ pairs $(\alpha, \beta) \in J \times J$ are connected in Γ

(loop would violate this)

(cont.) Write $e_\alpha = \frac{1}{\sqrt{(\alpha,\alpha)}}\alpha$ (unit vector)

$$v = \sum_{\alpha\in J} e_\alpha$$

Then $(v,v) = \sum_{\alpha\in J}(e_\alpha,e_\alpha) + \sum_{\alpha\neq\beta\in J}(e_\alpha,e_\beta)$

$(e_\alpha,e_\beta)\neq 0$ iff α and β are connected in Γ

then $-\frac{1}{2} \geq (e_\alpha,e_\beta)$ from the possibity of $n_{\alpha,\beta}$

So $(v,v) \leq |J| + \#\{(\alpha,\beta) : \alpha\sim\beta \text{ in } \Gamma\}\times(-\frac{1}{2})$

$(v,v)\geq 0$ implies $\#\{(\alpha,\beta) \text{ as above}\} < 2|J|$

Prop 3 each $\alpha\in\Pi$ is connected to at most 3 edges in Γ

Proof Step 1 $\#$ edges between α and β $= 4(e_\alpha,e_\beta)^2$

from $n_{\alpha,\beta}\, n_{\beta,\alpha} = \#$ edges.

Step 2 $\alpha\sim\beta,\ \alpha\sim\gamma \Rightarrow \beta\not\sim\gamma$ (Prop 2)

Step 3 $\sum_{\beta\neq\alpha} 4(e_\alpha,e_\beta)^2 < 4$ ($\overset{\Leftarrow}{\text{Step 1}}$ claim)

the e_β for $(e_\alpha,e_\beta)\neq 0$ are mutually orth. by Step 2

$\Rightarrow (e_\beta)_{\alpha\sim\beta}$ form an orthonormal basis of their span $V\subset E$.

$$\sum_{\beta\neq\alpha}(e_\alpha,e_\beta)^2 = \sum_{\beta\sim\alpha}(e_\alpha,e_\beta)^2 = (v,v) < 1.$$

for v = orth. proj. of e_α to V.