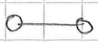
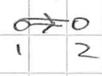
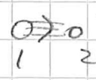
From root systems to Lie algebras.

(E, R) root system

$R = R_+ \sqcup R_-$ order, $\Pi \subset R_+$ simple pos roots

Def. The <u>Cartan matrix</u> of (E, R) is $A = (a_{ij})_{ij}$

$$a_{ij} = n_{\alpha_i, \alpha_j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad \text{for } \Pi = \{\alpha_1, \dots, \alpha_n\}.$$

Ex. A_2  $\leftrightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, B_2  $\leftrightarrow \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$, G_2  $\leftrightarrow \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Goal: construct cplx semisimple Lie alg $\mathfrak{g}_A$

Cartan subalg $\mathfrak{h}_A \subset \mathfrak{g}_A$ s.t.

1. the root system of $(\mathfrak{g}_A, \mathfrak{h}_A)$ is isom. to (E, R)
2. if cplx semisimple $\mathfrak{g}$, Cartan $\mathfrak{h} \subset \mathfrak{g}$ gives root sys (E, R) then $(\mathfrak{g}, \mathfrak{h}) \simeq (\mathfrak{g}_A, \mathfrak{h}_A)$ for A of (E, R)

How: mimick the relations between $E_\alpha, F_\alpha, H_\alpha$ in (semi) simple Lie algs.

Obvious ones

R1) $[E_\alpha, F_\alpha] = H_\alpha$ defining rel of H_α

R2) $[E_\alpha, F_\beta] = 0$ for $\alpha \neq \beta \in \Pi$ see below

R3) $[H_\alpha, H_\beta] = 0$ commutativity of $\mathfrak{h}$.

R4) $[H_{\alpha_i}, E_{\alpha_j}] = a_{ij} E_{\alpha_j}$, $[H_{\alpha_i}, F_{\alpha_j}] = -a_{ij} F_{\alpha_j}$

$$\alpha_j(H_{\alpha_i}) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad \text{(Oct 26 Prop. 1)}$$

R2: $\alpha \neq \beta \in \Pi \Rightarrow (\alpha, \beta) < 0 \Rightarrow \alpha + \beta \in R$ (Oct 27)

then $\alpha - \beta$ cannot be a root (otherwise $\alpha - \beta$ or $\beta - \alpha$ pos root, $\alpha = \beta + (\alpha - \beta), \cdots$)

cont.

$[E_\alpha, F_\beta) \in \mathfrak{g}_{\alpha-\beta} = 0$ by $F_\beta \in \mathfrak{g}_{-\beta}$.

Less-obvious rel (Serre relations)

R5) $\mathrm{Ad}_{E_{\alpha_i}}^{(1-a_{ij})}(E_{\alpha_j}) = 0 = \mathrm{Ad}_{F_{\alpha_i}}^{(1-a_{ij})}(F_{\alpha_j})$ $\forall i \neq j$)

Ex. $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. $\alpha_1 = \alpha_{12}$, $\alpha_2 = \alpha_{23}$ $a_{12} = -1$

R5 says $[E_{12}, [E_{12}, E_{23}]] = [E_{12}, E_{13}] = 0$

Prop. 1 Serre relations hold in simple Lie algs.

Proof. Consider the "α_i-string through α_j".

$\alpha_j + p\alpha_i, \alpha_j + (p-1)\alpha_i, \cdots, \alpha_j - q\alpha_i \subset R$

(p, q: max int s.t. this holds).

Step 1 $q = 0$

$\alpha_j - \alpha_i \notin R$ (see R4)

Step 2 $p = -a_{ij}$

$q - p = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} = a_{ij}$ (see Oct. 26)

& Step 1.

Step 3 $\mathrm{Ad}_{E_{\alpha_i}}^{1-a_{ij}}(E_{\alpha_j}) = 0$

$R \not\ni \alpha_j + (p+1)\alpha_i \underset{\text{Step 2}}{=} \alpha_j + (1-a_{ij})\alpha_i$

$\mathrm{Ad}_{E_{\alpha_i}}^{(1-a_{ij})}(E_{\alpha_j}) \in \mathfrak{g}_{\alpha_j + (1-a_{ij})\alpha_i} = 0$ $\square$

So, we should take ($n = |\Pi|$)

$\mathfrak{g}_A$ = Lie algebra with

- generators: E_i, F_i, H_i ($i = 1, \cdots, n$)

- relations: R1 - R5 for $E_{\alpha_i} = E_i$

$\mathfrak{h}_A$ = Lie subalg generated by $H_1, \cdots, H_n$

Prop 2 $\mathfrak{h}_A \subset \mathfrak{g}_A$ is a Cartan subalg.

Proof 1) ad_H ($H \in \mathfrak{h}_A$) are diagonalizable on $\mathfrak{g}_A$;

$\because$ assume $H = H_i$.

each E_j, F_j, H_j is an eigenvector for ad_H.

$\rightsquigarrow$ so are their brackets by Jacobi id.

$$[H, [X, Y]] = [[H, X], Y] + [X, [H, Y]]$$

2) $\mathfrak{h}_A$ is maximally commutative.

we can compute the weight of brackets of $E_i, F_i, \cdots$ (elem of E).

e.g. $[E_i, [E_j, F_j]]$ has weight $\alpha_i + \alpha_j - \alpha_j = \alpha_i$

such bracket commutes with all H_j

$\Leftrightarrow$ the weight λ satisfies $\lambda(H_j) = 0 \quad \forall j$

$\Leftrightarrow$ $\lambda + \alpha_j = 0$ $\quad \left(\lambda(H_j) = \dfrac{2(\lambda, \alpha_j)}{(\alpha_j, \alpha_j)} \right)$

$\Leftrightarrow$ $\lambda = 0$ $\quad$ (Π is a basis of E)

bracket of weight 0 reps an elem of $\mathfrak{h}_A$.

(only nontriv. cont $[E_i, F_i] = H_i$). $\square$

How to make sense of Lie alg defined by generators and relations. ;

$\rightsquigarrow$ consider the "free Lie algebra".

$U = \mathbb{C}\langle x_1, x_2, \cdots \rangle$

$\Delta : U \to U \otimes U$ $\quad \Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$

$\mathrm{Lie}(x_1, x_2, \cdots) = \{ a \in U : \Delta(a) = a \otimes 1 + 1 \otimes a \}$

bracket $[a, b] = ab - ba$

Prop 3. $\mathrm{Lie}(x_1, x_2, \cdots)$ is closed under this bracket.

Proof Suppose a, b satisfy $\Delta(a) = a \otimes 1 + 1 \otimes a$, etc.

$$\Delta([a,b]) = \Delta(ab - ba) = \Delta(a)\Delta(b) - \Delta(b)\Delta(a)$$
$$= [(a \otimes 1 + 1 \otimes a), (b \otimes 1 + 1 \otimes b)] \quad (\text{comm. br. in } U \otimes U)$$
$$= [a,b] \otimes 1 + 1 \otimes [a,b] \quad \boxtimes$$

Imposing relations

$R \subset \mathrm{Lie}\langle x_1, x_2, \cdots \rangle$ will specify relations among generators.

Ex. $x_1 = [x_2, x_3] \Leftrightarrow x_1 - [x_2, x_3] \in R$.

$\langle R \rangle$ = ideal generated by R
= span of $[\mathfrak{g}, [\mathfrak{g}, \cdots [\mathfrak{g}, R] \cdots]$
for $\mathfrak{g} = \mathrm{Lie}\langle x_1, x_2, \cdots \rangle$

Lie alg with generators $x_1, x_2, \cdots$, and relations
$R \subset \mathrm{Lie}\langle x_1, \cdots \rangle$: $\mathfrak{g} = \mathrm{Lie}\langle x_1, x_2, \cdots \rangle / \langle R \rangle$.

Ex. (E, R) root sys. $\rightsquigarrow$ Cartan mat A

$\rightsquigarrow \mathfrak{g}_A = \mathrm{Lie}\langle E_i, F_i, H_i \ (i = 1, \cdots, n) \rangle / \langle R \rangle$

$$R = \{ \begin{array}{ll} H_i - [E_i, F_i] & (\text{from R1}) \\ [E_i, F_j] \ (i \neq j) & (\text{from R2}) \\ [H_i, H_j] & (\text{from R3}) \\ [H_i, E_j] - a_{ij} E_j, \ [H_i, F_j] + a_{ij} F_j & (\text{from R4}) \\ \mathrm{ad}_{E_i}^{(1-a_{ij})}(E_j), \ \mathrm{ad}_{F_i}^{(1-a_{ij})}(F_j) & (\text{from R5}) \end{array} \}$$