Recall

root sys. $(E, R) \rightsquigarrow$ Cartan matrix $A = (a_{ij})_{ij}$

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad \text{for simpl pos. rts}$$

$\rightsquigarrow$ Lie alg $\mathfrak{g}_A$, Cartan subalg $\mathfrak{h}_A$ from

R1) $[E_i, F_i] = H_i$ R2) $[E_i, F_j] = 0$ $i \neq j$

R3) $[H_i, H_j] = 0$ R4) $[H_i, E_j] = a_{ij} E_j$

$[H_i, F_j] = -a_{ij} F_j$

R5) $\mathrm{ad}_{E_i}^{(1-a_{ij})}(E_j) = 0 = \mathrm{ad}_{F_i}^{(1-a_{ij})}(F_j)$

Want $\mathfrak{g}_A$ is semisimple

Put $\tilde{\mathfrak{g}} = \mathrm{Lie}\langle e_i, f_i, h_i \rangle / (R1, \dots, R4)$

$\tilde{n}_+$ = subalg generated by e_i $(i = 1, \dots, n)$

$\tilde{n}_-$ = ~ f_i ~

$x_{ij} = \mathrm{ad}_{e_i}^{(1-a_{ij})}(e_j)$, $y_{ij} = \mathrm{ad}_{f_i}^{(1-a_{ij})}(f_j)$

$\tilde{n}_+^0$ ideal of $\tilde{n}_+$ generated by x_{ij}

$\tilde{n}_-^0$ ~ y_{ij}

Intermed. goal $\tilde{n}_\pm^0 \triangleleft \tilde{\mathfrak{g}}$, $\mathfrak{g}_A = \tilde{\mathfrak{g}} / (\tilde{n}_+^0 + \tilde{n}_-^0)$

Prop. 1 $[\tilde{n}_-, x_{ij}] = 0$ (and $[\tilde{n}_+, y_{ij}] = 0$ by symmetry)

Proof. Step 1 $[f_k, x_{ij}] = 0$ for $k \neq i, j$

use $[f_k, e_i] = 0$, $[f_k, e_j] = 0$ and

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

Step 2 $[f_j, x_{ij}] = 0$

f_j comm. w/ e_i $\Rightarrow$ ad_{f_j} comm. w/ ad_{e_i}

$$\Rightarrow [f_j, x_{ij}] = \mathrm{ad}_{e_i}^{(1-a_{ij})} \mathrm{ad}_{f_j} e_j = -\mathrm{ad}_{e_i}^{(1-a_{ij})} h_j$$

$$= \mathrm{ad}_{e_i}^{-a_{ij}} (\underbrace{-[e_i, h_j]}_{a_{ji} e_i})$$

$a_{ji} = 0$ obvious, $a_{ji} \neq 0 \Rightarrow a_{ij} \neq 0 \Rightarrow \mathrm{ad}_{e_i}^{-a_{ij}}(e_i) = 0$

Step 3 $[f_i, x_{ij}] = 0$

$[\mathrm{ad}_{f_i}, \mathrm{ad}^k_{e_i}] = -k\, \mathrm{ad}^{k-1}_{e_i} (\mathrm{ad}_{h_i} + k - 1)$ from

$[\mathrm{ad}_{f_i}, \mathrm{ad}_{e_i}] = \mathrm{ad}_{[f_i, e_i]} = -\mathrm{ad}_{h_i}$

$[\mathrm{ad}_{h_i}, \mathrm{ad}_{e_i}] = \mathrm{ad}_{[h_i, e_i]} = 2\, \mathrm{ad}_{e_i}$

apply this for $k = 1 - a_{ij}$, use $\mathrm{ad}_{f_i}(e_j) = 0, \ldots$ ☐

Cor. $\tilde{\mathfrak{n}}^\circ_+ \triangleleft \tilde{\mathfrak{g}}$ (by symm $\tilde{\mathfrak{n}}^\circ_- \triangleleft \tilde{\mathfrak{g}}$).

e_i normalize $\tilde{\mathfrak{n}}^\circ_+$ by def

h_i ~ by $\mathrm{ad}_{h_i}(e_j) \in \mathbb{Z} e_j$.

f_i ~ by Prop 1

Next intermed goal :

recall Weyl group $W = \langle s_\alpha = \text{refl. across } \alpha^\perp : \alpha \in R \rangle \subset O(E)$

$W \curvearrowright \mathfrak{h}_A$ s.t. $s_{\alpha_i}(H_j) = H_j - a_{ij} H_i$

(induced action up to duality w/ $E \otimes \mathbb{C}$)

Want : this action is implemented by $\mathfrak{g}_A$

Def. $\theta_i = e^{\mathrm{ad}_{E_i}} e^{-\mathrm{ad}_{F_i}} e^{\mathrm{ad}_{E_i}} \in \mathrm{Aut}(\mathfrak{g}_A)$

Prop 2 This makes sense. ; $e^{\mathrm{ad}_{E_i}}(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{ad}^k_{E_i}(x)$

is finite sum for any $x \in \mathfrak{g}_A$ (same w/ F_i)

Proof. Step 1 x : one of generators.

$\mathrm{ad}^k_{E_i}(E_j) = 0$ for $k \geq 1 - a_{ij}$

$\mathrm{ad}^k_{E_i}(H_j) = 0$ for $k \geq 2$

$\mathrm{ad}^k_{E_i}(F_j) = 0$ for $k \geq 3$

Step 2 $x = [x_1, x_2]$ if x_i are OK

$\mathrm{ad}^k_{E_i}([x_1, x_2]) = \sum_{n=0}^{k} \binom{k}{n} [\mathrm{ad}^n_{E_i} x_1, \mathrm{ad}^{k-n}_{E_i} x_2]$

2021 - 11 - 09 no. 3

Motivation for $SL_2(\mathbb{C})$ $e^{\mathrm{ad}_E} e^{-\mathrm{ad}_F} e^{\mathrm{ad}_E} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

from $e^{\mathrm{ad}x} = \mathrm{Ad}_{e^x}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Prop 3 $\theta_i(H_j) = H_j - a_{ij} H_i = s_{\alpha_i}(H_j)$

Proof Direct computation.

Ex. Type A $i = 1, j = 2$ so $a_{ij} = -1$

$e^{\mathrm{ad}E_1} e^{-\mathrm{ad}F_1} e^{\mathrm{ad}E_1}(H_2) = e^{\mathrm{ad}E_1} e^{-\mathrm{ad}F_1}(H_2 + E_1)$

$= e^{\mathrm{ad}E_1}(H_2 + E_1 + F_1 + H_1 + -F_1)$

$= H_2 + E_1 + H_1 - E_1 = H_2 + H_1$

Th'm $\mathfrak{g}_A$ is semisimple.

Proof Step 1 Reduction : enough to prove

$\nexists$ nonzero comm. ideal of $\mathfrak{g}_A$.

$\mathrm{Rad}\,\mathfrak{g} \neq 0 \Rightarrow$ Take k s.t. $D^k(\mathrm{Rad}\,\mathfrak{g}) \neq 0$

but $D^{k+1}(\mathrm{Rad}\,\mathfrak{g}) = 0$

Step 2 $\mathfrak{b} \triangleleft \mathfrak{g} \Rightarrow \mathfrak{b} = (\mathfrak{b} \cap \mathfrak{h}_A) \oplus (\bigoplus_{\alpha \in \text{roots}} (\mathfrak{g}_A)_\alpha)$

ad_{H_i} $(i = 1, \cdots, n)$ simultaneously diagonalizable

with eigen dec. $\mathfrak{g}_A = \mathfrak{h}_A \oplus (\bigoplus_{\alpha\text{ roots}} (\mathfrak{g}_A)_\alpha)$

$\mathfrak{b}$ invariant under ad_{H_i}

Step 3 if $\mathfrak{b} \cap \mathfrak{h} \neq 0$ then $\exists i$ $E_i \in \mathfrak{b}$

$(\Rightarrow \theta_i(E_i) \underset{\text{dir. comp.}}{=} F_i \in \mathfrak{b} \Rightarrow \mathfrak{b}$ noncomm.)

For simplicity supp. (E, R) is irred

Fact. $W \curvearrowright E \otimes \mathbb{C}$ is an irred. rep.

$\Rightarrow W \curvearrowright \mathfrak{h}_A$ is also irred.

$\mathfrak{b} \cap \mathfrak{h}_A$ is W-inv $\Rightarrow \mathfrak{b} \cap \mathfrak{h}_A = \mathfrak{h}_A$

Step 4 $\mathfrak{b} \cap \mathfrak{g}_\alpha \neq 0 \Rightarrow \mathfrak{b}$ noncomm.

The action $W \curvearrowright \mathfrak{g}_A$ given by $s_{\alpha_i}(x) = \theta_i(x)$

(cont.) satisfies $s_\alpha(\mathfrak{g}_A)_\beta = (\mathfrak{g}_A)_{s_\alpha\beta}$

$[H, s_\alpha X] = s_\alpha[s_\alpha H, X] = \beta(s_\alpha H)\, s_\alpha X$

$X \in (\mathfrak{g}_A)_\beta$ in particular $s_\alpha((\mathfrak{g}_A)_\alpha) = (\mathfrak{g}_A)_{-\alpha}$

s_α is inner $\Rightarrow$ $s_\alpha(\mathfrak{h}) = \mathfrak{h}$. so $\mathfrak{h} \supset \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$ $\square$

Summary:

- $(E, R) \rightsquigarrow (\mathfrak{g}_A, \mathfrak{h}_A)$
- $(\mathfrak{g}, \mathfrak{h}) \rightsquigarrow (\mathfrak{h}_0^*, \underset{\text{roots}}{R(\mathfrak{g}, \mathfrak{h})})$ $\mathfrak{h}_0 = \langle H_\alpha : \alpha \in R\rangle_{\mathbb{R}}$

and

1) starting from $(\mathfrak{g}, \mathfrak{h})$, assoc Cartan mat A

$\underset{\text{semisimple}}{\mathfrak{g}_A} \xrightarrow{\varphi} \mathfrak{g}$, $E_i \mapsto E_{\alpha_i}$ etc. $\mathfrak{h}_A \xrightarrow{\sim} \mathfrak{h}$

$\Rightarrow$ φ is injective (hence isom.) also

2) starting from (E, R), root sys for $(\mathfrak{g}_A, \mathfrak{h}_A)$ is isom to (E, R) "by const"

Irreducibility of $W \curvearrowright E \otimes \mathbb{C}$ for irr. root sys.

Suppose $\mathcal{W} \subset E \otimes \mathbb{C}$ is invar under W

$\Leftrightarrow$ $s_\alpha \mathcal{W} = \mathcal{W}$ for all $\alpha \in R$.

$\Leftrightarrow$ $\alpha \in \mathcal{W}$ or $\alpha \perp \mathcal{W}$ for all $\alpha \in R$.

$R_0 = R \cap \mathcal{W}$, $R_1 = \{\alpha \in R : \alpha \perp \mathcal{W}\}$ is a decomp of R, $R_0 \perp R_1$

By irreducibility either $R_0 = \emptyset$ or $R_1 = \emptyset$

i.e. either $\mathcal{W} = 0$ or $\mathcal{W} = E \otimes \mathbb{C}$ $\square$