2021 - 11 - 10 no. 1

General structures for $\mathfrak{sl}_n \mathbb{C}$

Cartan subalg $\mathfrak{h} = \{ a_1 E_1 + \cdots + a_n E_n : a_1 + \cdots + a_n = 0 \}$

$E_i = E_{i,i}$ (H_i in the book)

linear dual $\mathfrak{h}^* = \langle L_1, \cdots, L_n \rangle_{\mathbb{C}} / (L_1 + \cdots + L_n)$.

$L_i(E_j) = \delta_{i,j}$

$\mathrm{ad}(a_1 E_1 + \cdots + a_n E_n)(E_{ij}) = [\sum_k a_k E_k, E_{ij}] = (a_i - a_j) E_{ij}$

Killing form B gives

$B(\sum a_i E_i, \sum b_i E_i) = \sum_{i \neq j} (a_i - a_j)(b_i - b_j)$

eigenval. of $\mathrm{ad}(\sum a_k E_k)\,\mathrm{ad}(\sum b_k E_k)$ on E_{ij}

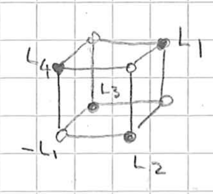
$= 2n \sum_{i=1}^{n} a_i b_i$

dual form B' of $B|_{\mathfrak{h}}$ is

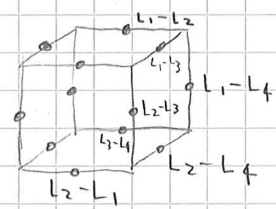
$B'(\sum a_i L_i, \sum b_i L_i) = \frac{1}{2n} \sum_i a_i b_i - \frac{1}{2n^2} \sum_{i,j} a_i b_j$

The case $n = 4$

$E = \mathfrak{h}_0^*$ 3-dim. with L_i ($i = 1, 2, 3, 4$)



$\rightsquigarrow$ roots



$\alpha_{ij} = L_i - L_j$ root for $E_{ij} \rightsquigarrow H_{\alpha_{ij}} = [E_{i,j}, E_{j,i}] = E_i - E_j$

$\Lambda_W = \{ \sum_i a_i L_i : a_i \in \mathbb{Z} \} / (\sum L_i)$

Λ_R = span of α_{ij} in Λ_W $= \{ \sum a_i L_i : \sum a_i = 0 \} \subset \Lambda_W$.

Exercise 15.7 $\Lambda_W / \Lambda_R \cong \mathbb{Z}/n\mathbb{Z}$

- $[L_i] = [L_1]$ in Λ_W/Λ_R : $L_i - L_1 \in \Lambda_R$
- $n[L_1] = 0$ in Λ_W/Λ_R : $n[L_1] = \sum_{i=1}^{n} [L_i] = [\sum_i L_i] = 0$

Order on R : $R_+ = \{ L_i - L_j : i < j \}$ upper triang.

$R_- = \{ L_i - L_j : i > j \}$ lower triang.

Weyl chamber $\mathcal{W} = \{ \sum a_i L_i : a_1 \geq a_2 \geq \cdots \geq a_n \} \subset E$

cone for $L_1, L_1 + L_2, \cdots, L_1 + \cdots + L_{n-1}$

Weyl group $\quad W = \langle s_\alpha : \alpha \in R \rangle \subset O(E)$

$\alpha = \alpha_{ij} \qquad s_\alpha(L_k) = \begin{cases} L_j & k=i \\ L_i & k=j \\ L_k & \text{otherwise} \end{cases}$

$\leftrightarrow$ perm $(ij) \in S_n$.

So $W \curvearrowright E$ is induced by perm. of $(L_i)_{i=1}^n$

$\mathcal{W}$ is a fundamental domain for this.

Th'm 1) $\forall \lambda \in \mathcal{W} \cap \Lambda_W$ $\exists!$ irred. rep (π, V) of $\mathfrak{sl}_n\mathbb{C}$ s.t. λ is its highest weight.

2) any irred. rep of $\mathfrak{sl}_n(\mathbb{C})$ is of this form

So irreps of $\mathfrak{sl}_n(\mathbb{C}) \equiv \mathcal{W} \cap \Lambda_W$.

Sketch for 2) : the weights of (π, V) is a union of orbits of W.

$\because s_\alpha(h) = \exp(\mathrm{ad}_{E_\alpha}) \exp(-\mathrm{ad}_{F_\alpha}) \exp(\mathrm{ad}_{E_\alpha})(h)$

μ weight of (π, V) $\qquad V_\mu \subset V$

$\Rightarrow \exp(E_\alpha)\exp(-F_\alpha)\exp(E_\alpha) V_\mu = V_{s_\alpha \mu}$

so $s_\alpha \mu$ is a weight of (π, V)

highest elem. of $\mathcal{W} \cap \{\text{weights of } (\pi, V)\}$ will be the hieghest weight of (π, V). $\boxtimes$

More on reps of $\mathfrak{sl}_4\mathbb{C}$

$V = \mathbb{C}^4$ defining rep.

- weights $L_1, L_2, L_3, L_4 = -(L_1+L_2+L_3)$.

highest weight L_1

Symmetric power $\mathrm{Sym}^k V$

space of lin combs of $v_1 \cdots v_k$ $(v_i \in V)$

$v_{i_1} \cdots v_{i_k} = v_1 \cdots v_k$ for perm $(i_1, \cdots, i_k)$

(cont.) concretely: subsp. of $V^{\otimes k}$ consisting of the S_k-invariant vectors

$$v_1 \cdots v_k \leftrightarrow \frac{1}{k!} \sum_{(i_1, \cdots, i_k)} v_{i_1} \otimes \cdots \otimes v_{i_k}$$

Basis of $\mathrm{Sym}^k V$: $\underbrace{e_1 \cdots e_1}_{a_1} \underbrace{e_2 \cdots e_2}_{a_2} \underbrace{e_3 \cdots e_3}_{a_3} \underbrace{e_4 \cdots e_4}_{a_4}$

$a_1 + a_2 + a_3 + a_4 = k$

corresp. weight. $a_1 L_1 + a_2 L_2 + a_3 L_3 + a_4 L_4$

$$(= (2a_1 + a_2 + a_3 - k) L_1 + (a_1 + 2a_2 + a_3 - k) L_2 + (a_1 + a_2 + 2a_3 - k) L_3)$$

highest weight: $k L_1$

action of $\mathfrak{sl}_4 \mathbb{C}$ (rather $\mathfrak{n}_- = \langle E_{i,j} \quad i > j \rangle$)

$E_{2,1}$: $e_1^{a_1} e_2^{a_2} e_3^{a_3} e_4^{a_4} \mapsto a_1 e_1^{a_1 - 1} e_2^{a_2+1} e_3^{a_3} e_4^{a_4}$

$E_{3,2}$: $\sim \mapsto a_2 e_1^{a_1} e_2^{a_2 - 1} e_3^{a_3+1} e_4^{a_4}$

etc.

So $E_{4,3}^{a_4} E_{3,2}^{a_3+a_4} E_{2,1}^{a_2+a_3+a_4} e_1^k = a_4 (a_3 + a_4)(a_2 + a_3 + a_4)$
$\times e_1^{a_1} e_2^{a_2} e_3^{a_3} e_4^{a_4}$

Prop. $\mathrm{Sym}^k V$ is irreducible $\mathfrak{sl}_4(\mathbb{C})$-rep.

Proof take $\mathrm{Sym}^k V = W \oplus W'$ for $\mathfrak{sl}_4(\mathbb{C})$-inv subspaces W, W'

$e_1^k + \cdots \in W$ or $e_1^k + \cdots \in W'$

By the action of $\mathfrak{h}$, we get $e_1^k \in W$ or $e_1^k \in W'$. By above obs. $e_1^k \in W \Rightarrow W = V$ ☒

Exterior power $\Lambda^k V$

space of lin. combs $v_1 \wedge \cdots \wedge v_k$ $(v_i \in V)$

$v_1 \wedge v_2 = - v_2 \wedge v_1$ (so $\Lambda^4 V = 0$)

Basis $e_{i_1} \wedge \cdots \wedge e_{i_k}$ $i_1 < i_2 < \cdots < i_k$

corr. weight $L_{i_1} + \cdots + L_{i_k}$

$\Lambda^3 V$ is 4-dim. weights

highest: $L_1 + L_2 + L_3 = -L_4$, $L_1 + L_2 + L_4 = -L_3$, $L_1 + L_3 + L_4 = -L_2$

$L_2 + L_3 + L_4 = -L_1$

$\Lambda^3 V \simeq V^*$ (as rep.)

$\Lambda^2 V$ is 6-dim weights, weights $L_i + L_j$ $(i \neq j)$

this is a single orbit of $W = S_3$

highest elem $L_1 + L_2$

$\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(\Lambda^2 V) \otimes \mathrm{Sym}^c(\Lambda^3 V)$ has highest

weight $aL_1 + b(L_1 + L_2) + c(L_1 + L_2 + L_3) = \omega_{a,b,c}$

$\mathfrak{sl}_4(\mathbb{C})$-span of the highest weight vector

$=$ irred. rep. of highest weight $\omega_{a,b,c} \in W$