$\mathfrak{g}$ (semi) simple complex Lie alg, G corr. Lie grp.
$\mathfrak{h} < \mathfrak{g}$ Cartan subalg. R root, $R = R_+ \amalg R_-$.
so $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right), \dots$

Def. put $\mathfrak{b} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \right)$ solvable Lie subalg
(with $\mathcal{D}(\mathfrak{b}) = \mathfrak{n}_+ = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$ nilpot, ...)
and $B < G$ corresponding subgroup
call it a Borel subgroup

Example $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. $\mathfrak{b} = \left\{ \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \text{ with } \mathrm{tr} = 0 \right\}$

$B = \left\{ \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \text{ with } \prod \text{diags} = 1 \right\}$

Def. G/B is called a/the (full) flag variety of G.

Fact G/B is a closed smooth manifold
(in fact complex projective variety)

Ex. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ B as above is the stabilizer subgroup for the action $SL_n(\mathbb{C}) \curvearrowright \{ V_1 < \dots < V_{n-1} < \mathbb{C}^n \}$, $\dim V_k = k$

B fixes $V_\bullet$ for $V_k = \left\langle \begin{bmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \end{bmatrix} \right\rangle k$.

then $G/B = SU(n)/T$. $T = \left\{ \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix} \right\}$.

similar arg. holds with a compact form of G in general.

$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \rightsquigarrow G/B = S^2 = \mathbb{P}^1(\mathbb{C})$.

Some structures from algebraic topology.

X: finite simplicial complex (closed sm. mfd is enough)
$f: X \to X$ cont. map
Lefschetz fixed point formula)

(cont.) $\#(\text{fixed points of } f) \geq \Lambda_f = \mathrm{Tr}\, f^*|_{H^{ev}(X)} - \mathrm{Tr}\, f^*|_{H^{odd}(X)}$

$$H^{ev}(X) = \bigoplus_{k=0}^{\infty} H^{2k}(X), \quad H^{odd}(X) = \bigoplus_{k=0}^{\infty} H^{2k+1}(X)$$

Application : uniqueness of Cartan algebras.

Th'm. $\mathfrak{h}, \mathfrak{h}' < \mathfrak{g}$ Cartan subalgs.

then $\exists$ automorphism φ of $\mathfrak{g}$ s.t. $\varphi(\mathfrak{h}') = \mathfrak{h}$

Outline.

Step 1. $\exists X \in \mathfrak{h}$ "generic" in the sense $\alpha(H) \neq 0$

for all root α of $(\mathfrak{g}, \mathfrak{h})$ s.t.

$\mathfrak{h} = \{ Y \in \mathfrak{g} : [X, Y] = 0 \}$ (centralizer of X)

Step 2 when $X' \in \mathfrak{h}'$ characterizes $\mathfrak{h}'$ as above

$\exists g \in G$ s.t. $\mathrm{Ad}_g(X') \in \mathfrak{b}$.

$\because$ look at the left multiplication by $e^{X'}$ on G/B.

- homotopic to identity by $e^{tX'}$ $0 \leq t \leq 1$

- $H^{odd}(G/B) = 0$; G/B has CW-cplx str. with only even-dim cells.

$\Rightarrow \Lambda_f = \dim H^{ev}(G/B)$ ($= |\text{Weyl group}|$) > 0

So $e^{X'}$ has a fixed point, say $g^{-1}B$

$$e^{X'} g^{-1} B = g^{-1} B \Leftrightarrow g e^{X'} g^{-1} \in B.$$

Taking log, $gX'g^{-1} \in \mathfrak{b}$ (start with small enough X')

Step 3. $\mathrm{Ad}_g(X') \in \mathfrak{b} \Rightarrow \mathrm{Ad}_g(X') \in \mathfrak{h}$.

uniqueness of semisimple-nilpotent decomposition.