1b. We know that

$$(\mathrm{Ind}_I^G \omega)(g_1) W_{g_2 I} \subset W_{g_1 g_2 I}.$$

In particular, if $h \in H$, $g \in G$, then

$$(\mathrm{Ind}_I^G \omega)(h) W_{gI} \subset W_{hgI} = W_{g(g^{-1}hg)I} = W_{gI},$$

since $g^{-1}hg \in H \subset I$.

Similarly, as $W_{gI} = (\mathrm{Ind}_I^G \omega)(g) W_I$, we have, for $h \in H$, on W_{gI}

$$\begin{aligned}
(\mathrm{Ind}_I^G \omega)(g^{-1})(\mathrm{Ind}_I^G \omega)(h) &= (\mathrm{Ind}_I^G \omega)(g^{-1}hg)(\mathrm{Ind}_I^G \omega)(g^{-1}) \\
&= \omega(g^{-1}hg)(\mathrm{Ind}_I^G \omega)(g^{-1}) \\
&\qquad (\text{we identify } W_I \text{ with } V_\omega) \\
&= (\mathrm{Res}_H^I \omega)(h)^g (\mathrm{Ind}_I^G \omega)(g^{-1}),
\end{aligned}$$

so $(\mathrm{Ind}_I^G \omega)(g^{-1})$ intertwines

$\mathrm{Res}_H^G \mathrm{Ind}_I^G \omega$ with $(\mathrm{Res}_H^I \omega)^g$.

Finally, as $\mathrm{Res}_H^I \omega \sim m\pi$, we have $(\mathrm{Res}_H^I \omega)^g \sim m\pi^g$.

1c. We need to show that

$$\dim \mathrm{Mor}(\mathrm{Ind}_I^G \omega, \mathrm{Ind}_I^G \omega) = 1,$$

equivalently, by Frobenius reciprocity,

$$\dim \mathrm{Mor}(\omega, \mathrm{Res}_I^G \mathrm{Ind}_I^G \omega) = 1.$$

Assume $T \in \mathrm{Mor}(\omega, \mathrm{Res}^G_I \mathrm{Ind}^G_I \omega)$.

Then T intertwines also $\mathrm{Res}^I_H \omega$ ~~and~~ with $\mathrm{Res}^G_H \mathrm{Ind}^G_I \omega$. But by (b) we know that

$$\mathrm{Res}^G_H \mathrm{Ind}^G_I \omega \sim \bigoplus_{g \in G/I} m\pi^g .$$

As $\pi^g \not\sim \pi$ for $g \notin I$, we conclude that the image of T must be contained in W_I. Identifying again W_I with ~~...~~ V_ω we thus see that $T \in \mathrm{Mor}(\omega, \omega) = \mathbb{C}1$

2a. By exercise 2 from the second set it suffices to show that the action of G on $\mathbb{F}_q$ is doubly transitive.

Alternatively, one can compute the character of the representation and check that $(\chi_\pi, \chi_\pi) = 1$.

2c. From the first exercise we can conclude that

$$\mathrm{Res}^G_H \mathrm{Ind}^G_H \chi \sim \bigoplus_{a \in \mathbb{F}_q^\times} \chi(a^{-1} \cdot)$$
$$= \bigoplus_{\substack{\chi' \in \hat{\mathbb{F}}_q \\ \chi' \neq \varepsilon}} \chi' .$$

By the orthogonality relations we have

$\sum_{a\in\mathbb{F}_4} \chi'(a)=0$ for all $\chi'\neq\varepsilon$, so χ' lies in the space of functions in (a).

This gives an indication how to define an intertwiner $T: \mathrm{Ind}_H^G \chi \to \bar{\pi}$:

For $f: G\to\mathbb{C}$ s.t. $f\left(g\begin{pmatrix}1&b\\0&1\end{pmatrix}\right)=\overline{\chi(b)}\,f(g)$, define $Tf:\mathbb{F}_4\to\mathbb{C}$ by

$$Tf=\sum_{a\in\mathbb{F}_4^\times} f\begin{pmatrix}a&0\\0&1\end{pmatrix}\overline{\chi(a^{-1}\cdot)}.$$

The map T is injective, since $\chi(a^{-1}\cdot)$ $(a\in\mathbb{F}_4^\times)$ are linearly independent. Let us check that it is indeed an intertwiner:

$$\begin{aligned}
T\left(\mathrm{Ind}_H^G\chi\right)\begin{pmatrix}a_0&b_0\\0&1\end{pmatrix}f
&=Tf\left(\begin{pmatrix}a_0^{-1}&-a_0^{-1}b_0\\0&1\end{pmatrix}\cdot\right)\\
&=\sum_{a\in\mathbb{F}_4^\times} f\left(\begin{pmatrix}a_0^{-1}&-a_0^{-1}b_0\\0&1\end{pmatrix}\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)\overline{\chi(a^{-1}\cdot)}\\
&=\sum_{a\in\mathbb{F}_4^\times} f\left(\begin{pmatrix}a_0^{-1}a&0\\0&1\end{pmatrix}\begin{pmatrix}1&-a^{-1}b_0\\0&1\end{pmatrix}\right)\overline{\chi(a^{-1}\cdot)}\\
&=\sum_{a\in\mathbb{F}_4^\times} f\begin{pmatrix}a_0^{-1}a&0\\0&1\end{pmatrix}\overline{\chi(a^{-1}(\cdot-b_0))}\\
&=\sum_{a\in\mathbb{F}_4^\times} f\begin{pmatrix}a&0\\0&1\end{pmatrix}\overline{\chi(a^{-1}a_0^{-1}(\cdot-b_0))}
\end{aligned}$$

$$= \pi\begin{pmatrix} a_0 & b_0 \\ 0 & 1 \end{pmatrix} T f,$$

since by definition

$$\left(\tilde\pi\begin{pmatrix} a_0 & b_0 \\ 0 & 1 \end{pmatrix}\tilde f\right)(v) = \tilde f\left(a_0^{-1}(v-b_0)\right).$$

3b. As $\bar\pi$ and θ are irreducible, we have

$$\bar\pi(\mathbb{C}G) = \mathrm{End}(V_{\bar\pi}),\ \theta(\mathbb{C}H) = \mathrm{End}(V_\theta).$$

hence

$$(\bar\pi\otimes\theta)(\mathbb{C}(G\times H)) = (\bar\pi\otimes\theta)(\mathbb{C}G\otimes\mathbb{C}H)$$

$$= \mathrm{End}(V_{\bar\pi})\otimes\mathrm{End}(V_\theta)$$

$$= \mathrm{End}(V_{\bar\pi}\otimes V_\theta),$$

which implies that $\bar\pi\otimes\theta$ is irreducible.

Alternatively,

$$(\chi_{\bar\pi\otimes\theta}, \chi_{\bar\pi\otimes\theta}) = (\chi_{\bar\pi},\chi_{\bar\pi})(\chi_\theta,\chi_\theta) = 1.$$

(But ~~___~~ the first argument works for infinite groups as well.)

3c. If $T\in \mathrm{Mor}(\bar\pi, \mathrm{Res}^{G\times H}_H\,\omega)$, $h\in H$, then $\omega(h)T\in \mathrm{Mor}(\bar\pi, \mathrm{Res}^{G\times H}_H\,\omega)$, since $\omega(h)\omega(g) = \omega(g)\omega(h)$ $\forall g\in G$. Hence $\tilde\theta$ is well-defined. Then $S: V_{\bar\pi}\otimes V_\theta\to V$, $v\otimes T\mapsto Tv$, is nonzero and intertwines $\bar\pi\otimes\theta$ with ω. As $\bar\pi\otimes\theta$ and ω are irreducible, we conclude that $\bar\pi\otimes\theta\sim\omega$.