UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4301 — Partial Differential Equations

Day of examination: Thursday 28 November 2019

Examination hours: 09:00 – 13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weight 15%)

Consider the following first-order PDEs. For each PDE:

- If the problem has a solution,
  - use the method of characteristics to solve it,
  - verify that the formula that you have found is correct.
- If the problem does not have a solution, explain why.
- In either case, draw some of the characteristic curves.

1a

$$\begin{cases} u_x - u_y = 0 & \text{for } x, y \in (0,1) \\ u(0,y) = y & \text{for } y \in [0,1] \\ u(x,1) = 1 - x^2 & \text{for } x \in [0,1]. \end{cases} \tag{1}$$

1b

$$\begin{cases} tu_t + 2u_x = 0 & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = \sin(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{2}$$

Problem 2 The wave equation (weight 10%)

Let $T > 0$. Find the general solution of the *backwards* problem

$$\begin{cases} u_{tt} = u_{xx} & \text{for } x \in \mathbb{R},\ t \in (0,T) \\ u(x,T) = g(x) & \text{for } x \in \mathbb{R} \\ u_t(x,T) = h(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{3}$$

Is the solution unique?

(Continued on page 2.)

Problem 3 A conservation law (weight 5%)

Find a weak solution of the problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = \begin{cases} 3 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \end{cases} \tag{4}$$

where $f(u) = u^4$. Does your solution satisfy the entropy condition?

Problem 4 Duhamel's principle (weight 15%)

4a

Verify that $u(x,t) = e^{-t}g(x - bt)$ solves the advection-reaction equation

$$\begin{cases} u_t + b \cdot Du = -u & \text{for } x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^d \end{cases} \tag{5}$$

where $b \in \mathbb{R}^d$ is a given vector and $g \in C^1(\mathbb{R}^d)$ is a given function.

4b

Use Duhamel's principle to find the solution of the corresponding nonhomogeneous equation

$$\begin{cases} u_t + b \cdot Du = -u + f & \text{for } x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = g & \text{for } x \in \mathbb{R}^d \end{cases} \tag{6}$$

for a function $f \in C(\mathbb{R}^d \times [0,\infty))$. As in **4a**, verify that your answer is indeed a solution of (6).

Problem 5 Harmonic functions (weight 30%)

Let $U \subset \mathbb{R}^d$ be open, bounded and connected, and let $u \in C^\infty(\mathbb{R}^d)$.

5a

Show that if u is harmonic in U then $D^\alpha u$ is harmonic for any multi-index α.

5b

Conversely, show that if u_{x_i} is harmonic in U for every $i = 1, \ldots, n$, then u satisfies

$$\Delta u = a \qquad \text{in } U$$

for some constant $a \in \mathbb{R}$.

(Continued on page 3.)

5c

Assume u satisfies

$$-\Delta u = f \qquad \text{in } U \tag{7}$$

for a polynomial f of degree $k \in \mathbb{N}$. Prove the mean value formula

$$D^\alpha u(x) = \fint_{B(x,r)} D^\alpha u(y)\,dy \tag{8}$$

for any multi-index $|\alpha| > k$. For what $x \in U$ and $r > 0$ is the formula valid? *(You may use the mean value formula for harmonic functions.)*

5d

Use **5c** to prove the following maximum principle for any multi-index $|\alpha| > k$:

$$D^\alpha u(x) \leqslant \max_{\partial U} D^\alpha u \qquad \forall\, x \in U. \tag{9}$$

Problem 6 (weight 25%)

~~Let $U \subset \mathbb{R}^n$ be open, bounded and connected.~~ Consider the advection-diffusion problem

$$\begin{cases} u_t + f(u)_x = \varepsilon u_{xx} & \text{for } x \in (0,1),\ t \in (0,T] \\ u(0,t) = u(1,t) = 0 & \text{for } t \in (0,T] \\ u(x,0) = g(x) & \text{for } x \in (0,1) \end{cases} \tag{10}$$

where $f(u) = u^3$, $\varepsilon > 0$ is a given number and $g \in C([0,1])$ satisfies $g(0) = g(1) = 0$. Let $u \in C^2((0,1)\times(0,T]) \cap C([0,1]\times[0,T])$ be a solution of (10).

6a Energy method

Prove that $E[u](t) := \int_0^1 u(x,t)^2\,dx$ decreases over time.

6b Maximum principle

Prove that $\min_{y\in[0,1]} g(y) \leqslant u(x,t) \leqslant \max_{y\in[0,1]} g(y)$ for every $x \in [0,1]$, $t \in [0,T]$.

Hint: Prove the result for $v^\delta(x,t) = u(x,t) - \delta t$ for some $\delta > 0$ first. What equation does v^δ satisfy?

6c Uniqueness

Unlike for the heat equation, we cannot apply the results in **6a** or **6b** to prove uniqueness of the solution of (10). Why not?

THE END