# UNIVERSITY OF OSLO

## Faculty of mathematics and natural sciences

Exam in: MAT4301 — Partial Differential Equations

Day of examination: Thursday 28 November 2019

Examination hours: 09:00 – 13:00

This problem set consists of 9 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

## Problem 1 (weight 15%)

Consider the following first-order PDEs. For each PDE:

- If the problem has a solution,
  - use the method of characteristics to solve it,
  - verify that the formula that you have found is correct.
- If the problem does not have a solution, explain why.
- In either case, draw some of the characteristic curves.

**1a**

$$\begin{cases} u_x - u_y = 0 & \text{for } x, y \in (0,1) \\ u(0,y) = y & \text{for } y \in [0,1] \\ u(x,1) = 1 - x^2 & \text{for } x \in [0,1]. \end{cases} \tag{1}$$

**Solution:** The equations of characteristics are

$$\dot{x} = 1, \qquad \dot{y} = -1, \qquad \dot{z} = 0,$$

with $x(0) = x^0$, $y(0) = y^0$, $z(0) = z^0$ and $(x^0, y^0) \in \Gamma = \{(x,y) \in \partial U : x = 0 \text{ or } y = 1\}$, where $U = (0,1)^2$. (Since these equations are independent of $p = Du$, we do not need the equation for $\dot{p}$.) The solution is

$$x(s) = x^0 + s, \qquad y(s) = y^0 - s, \qquad z(s) = z^0.$$

We have $x^0 = 0$ if and only if $y(s) \leqslant 1 - x(s)$ for all $s$, and $y^0 = 1$ if and only if $y(s) \geqslant 1 - x(s)$ for all $s$. Hence, for an arbitrary point $(x,y) \in U$ we have $(x(s), y(s)) = (x,y)$ if and only if $s = x$, $x^0 = 0$, $y^0 = x + y$ when $y \leqslant 1 - x$, and $y^0 = 1$, $s = 1 - y$, $x^0 = x + y - 1$ when $y \geqslant 1 - x$.

*(Continued on page 2.)*

This leads to the solution

$$u(x,y) = \begin{cases} y^0 = x + y & \text{if } y \leqslant 1 - x \\ 1 - (x^0)^2 = 1 - (x+y-1)^2 & \text{if } y \geqslant 1 - x. \end{cases}$$

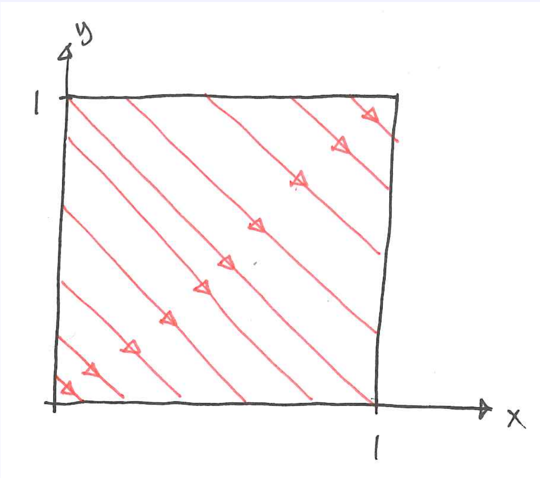
(Note that the solution is continuous across $y = 1 - x$.) We verify that $u$ solves the PDE:

$$u_x(x,y) - u_y(x,y) = \begin{cases} 1 - 1 = 0 & \text{if } y \leqslant 1 - x \\ -2(x+y-1) + 2(x+y-1) = 0 & \text{if } y \geqslant 1 - x \end{cases}$$

and the boundary conditions:

$$u(0,y) = 0 + y = y, \qquad u(x,1) = 1 - (x+1-1)^2 = 1 - x^2.$$

A sketch of the characteristics and their direction (direction of increasing $s$) is shown below.




**1b**

$$\begin{cases} tu_t + 2u_x = 0 & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = \sin(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{2}$$

**Solution:** Write the problem as

$$\begin{cases} F(Du(X), u(X), X) = 0 & \text{in } U \\ u = g & \text{on } \Gamma \end{cases}$$

where $U = \mathbb{R}\times\mathbb{R}_+$, $\Gamma = \mathbb{R}\times\{0\}$, $X = \begin{pmatrix} x \\ t \end{pmatrix}$ and $F(p,z,X) = p\cdot\begin{pmatrix} 2 \\ t \end{pmatrix}$. We have $X^0 \in \Gamma$ if and only if $t^0 = 0$, and a point $(p^0, z^0, X^0)$ is admissible if and only if

$$t^0 = 0, \quad z^0 = g(X^0) = e^{x^0}, \quad p_1^0 = g_x(X^0) = \cos(x^0), \quad F(p^0, z^0, X^0) = 0.$$

But $F(p^0, z^0, X^0) = 2p_1^0 = 2\cos(x^0)$, which is only zero for certain

(Continued on page 3.)

choices of $x^0$, namely $x^0 = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$.

The noncharacteristic condition at $(p^0, z^0, X^0)$ reads

$$0 \neq \nu(X^0) \cdot D_p F(p^0, z^0, X^0) = -t^0 = 0,$$

so the condition is never satisfied. Hence, the method of characteristics is not applicable.

The equations for characteristics are

$$\dot{t}(s) = t(s), \qquad \dot{x}(s) = 2, \qquad \dot{z}(s) = 0$$

with $t(0) = t^0 = 0$, $x(0) = x^0$, $z(0) = z^0 = x^0$. We find that

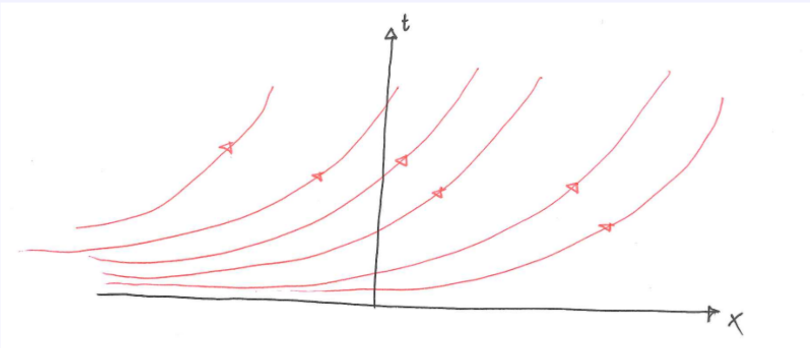
$$x(s) = x^0 + 2s, \qquad t(s) = t^0 e^s.$$

Hence, $t^0 = 0$ would yield $t(s) \equiv 0$, and the characteristics never enter the domain $U$.

Without applying the boundary condition $t^0 = 0$ we get the relations

$$s = \log(ct) \qquad \Rightarrow \qquad x(t) = x^0 + 2\log(ct) \qquad \text{for } c = \frac{1}{t^0} \in \mathbb{R}$$

which can be seen in the figure below.




## Problem 2 The wave equation (weight 10%)

Let $T > 0$. Find the general solution of the *backwards* problem

$$\begin{cases} u_{tt} = u_{xx} & \text{for } x \in \mathbb{R},\ t \in (0,T) \\ u(x,T) = g(x) & \text{for } x \in \mathbb{R} \\ u_t(x,T) = h(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{3}$$

Is the solution unique?

**Solution (approach 1):** We define $v = u_t - u_x$ and obtain the system of transport equations

$$\begin{cases} v_t + v_x = 0 & \text{for } x \in \mathbb{R},\ t \in (0,T) \\ v(x,T) = h(x) - g'(x) & \text{for } x \in \mathbb{R} \\ u_t - u_x = v & \text{for } x \in \mathbb{R},\ t \in (0,T) \\ u(x,T) = g(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

*(Continued on page 4.)*

The characteristic for $v$ going through a point $(x,t)$ is $s \mapsto (x+s, t+s)$ and since $v$ is constant along this curve we get for $s=0$ and $s=T-t$

$$v(x,t) = v(x+T-t, T) = h(x+T-t) - g'(x+T-t).$$

Likewise, the characteristics for $u$ are $s \mapsto (x-s, t+s)$, so integrating the equation along the characteristic over $s \in (0, T-t)$ gives

$$\begin{aligned} u(x,t) &= u(x-T+t, T) - \int_0^{T-t} v(x-s, t+s)\, ds \\ &= g(x-T+t) - \int_0^{T-t} h(x-s+T-t-s) - g'(x-s+T-t-s)\, ds \\ &= -\frac{1}{2}\int_{x-T+t}^{x+T-t} h(y)\, dy + \frac{1}{2}\left(g(x-T+t) + g(x+T-t)\right). \end{aligned}$$

Since every step in the above calculation was *necessary*, the solution must be unique.

**Solution (approach 2):** Let $v(x,t) = u(x, T-t)$ for $x \in \mathbb{R}$, $t \in [0,T]$. Then

$$\begin{cases} v_{tt} = v_{xx} & \text{for } x \in \mathbb{R},\ t \in (0,T) \\ v(x,0) = g(x) & \text{for } x \in \mathbb{R} \\ v_t(x,0) = -h(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Hence, d'Alembert's formula gives

$$v(x,t) = -\frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy + \frac{1}{2}\left(g(x-t) + g(x+t)\right).$$

Since $u(x,t) = v(x, T-t)$ we conclude that

$$u(x,t) = -\frac{1}{2}\int_{x-T+t}^{x+T-t} h(y)\, dy + \frac{1}{2}\left(g(x-T+t) + g(x+T-t)\right).$$

Since the solution $v$ is unique, the solution $u$ is automatically unique.

## Problem 3 A conservation law (weight 5%)

Find a weak solution of the problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = \begin{cases} 3 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \end{cases} \tag{4}$$

where $f(u) = u^4$. Does your solution satisfy the entropy condition?

**Solution:** We make the ansatz

$$u(x,t) = \begin{cases} 3 & \text{for } x < st \\ 1 & \text{for } x > st \end{cases}$$

*(Continued on page 5.)*

for some $s \in \mathbb{R}$. The Rankine–Hugoniot condition demands that $u$ is a classical solution on either side of the discontinuity (which it is, since constants solve (4)), and that the jump condition is satisfied:

$$f(3) - f(1) = s(3-1) \qquad \Leftrightarrow \qquad s = \frac{3^4 - 1^4}{2} = 40.$$

The flux function is convex, so the entropy condition reduces to the condition that

$$f'(u^L) \geqslant s \geqslant f'(u^R) \qquad \Leftrightarrow \qquad 4 \cdot 3^3 = 108 \geqslant 40 \geqslant 4 \cdot 1^3 = 4,$$

which is clearly true. Hence, the function

$$u(x,t) = \begin{cases} 3 & \text{for } x < 40t \\ 1 & \text{for } x > 40t \end{cases}$$

is the entropy solution of (4).

## Problem 4 Duhamel's principle (weight 15%)

### 4a

Verify that $u(x,t) = e^{-t} g(x - bt)$ solves the advection-reaction equation

$$\begin{cases} u_t + b \cdot Du = -u & \text{for } x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^d \end{cases} \tag{5}$$

where $b \in \mathbb{R}^d$ is a given vector and $g \in C^1(\mathbb{R}^d)$ is a given function.

**Solution:** We have $u(x,0) = e^0 g(x-0) = g(x)$, and

$$u_t(x,t) = -e^{-t} g(x-bt) + e^{-t} Dg(x-bt) \cdot (-b) = -u(x,t) - b \cdot Du(x,t),$$

so $u$ solves the PDE.

### 4b

Use Duhamel's principle to find the solution of the corresponding nonhomogeneous equation

$$\begin{cases} u_t + b \cdot Du = -u + f & \text{for } x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) = g & \text{for } x \in \mathbb{R}^d \end{cases} \tag{6}$$

for a function $f \in C(\mathbb{R}^d \times [0,\infty))$. As in **4a**, verify that your answer is indeed a solution of (6).

**Solution:** Let first $v = v(x,t;s)$ for $t > s$ be the solution of

$$\begin{cases} v_t + b \cdot Dv = -v & \text{for } x \in \mathbb{R}^d,\ t > s \\ v(x,s) = f(x,s) & \text{for } x \in \mathbb{R}^d \end{cases}$$

*(Continued on page 6.)*

which we, by **4a**, can write as

$$v(x,t;s) = e^{-(t-s)} f(x - b(t-s), s).$$

Now define

$$v(x,t) = \int_0^t v(x,t;s)\,ds = \int_0^t e^{-(t-s)} f(x - b(t-s), s)\,ds.$$

Then

$$\begin{aligned} v_t(x,t) &= e^{-(t-s)} f(x - b(t-s), s)\Big|_{s=t} \\ &\quad + \int_0^t -e^{-(t-s)} f(x - b(t-s), s) + e^{-(t-s)} Df(x - b(t-s), s) \cdot (-b)\,ds \\ &= f(x,t) - v(x,t) - b \cdot \int_0^t e^{-(t-s)} Df(x - b(t-s), s)\,ds \\ &= f(x,t) - v(x,t) - b \cdot Dv(x,t), \end{aligned}$$

so $v$ satisfies the inhomogeneous PDE, and $v(x,0) = 0$. To satisfy the initial data in (6) we add the solution from **4a** to get

$$u(x,t) = e^{-t} g(x - bt) + \int_0^t e^{-(t-s)} f(x - b(t-s), s)\,ds.$$

Then $u(x,0) = e^0 g(x-0) + 0 = g(x)$, and since $u$ is the sum of the homogeneous and inhomogeneous equations, it solves the inhomogeneous equation.

## Problem 5 Harmonic functions (weight 30%)

Let $U \subset \mathbb{R}^d$ be open, bounded and connected, and let $u \in C^\infty(\mathbb{R}^d)$.

### 5a

Show that if $u$ is harmonic in $U$ then $D^\alpha u$ is harmonic for any multi-index $\alpha$.

**Solution:**

$$\Delta(D^\alpha u)(x) = D^\alpha(\Delta u)(x) = 0.$$

### 5b

Conversely, show that if $u_{x_i}$ is harmonic in $U$ for every $i = 1, \ldots, n$, then $u$ satisfies

$$\Delta u = a \qquad \text{in } U$$

for some constant $a \in \mathbb{R}$.

**Solution:** We have

$$0 = \Delta(u_{x_i})(x) = (\Delta u)_{x_i}(x) \qquad \forall\, i = 1, \ldots, n.$$

*(Continued on page 7.)*

Hence, for every $i$, the function $\Delta u(x)$ is constant in the $i$th variable – in other words, $\Delta u$ is constant.

### 5c

Assume $u$ satisfies

$$-\Delta u = f \qquad \text{in } U \tag{7}$$

for a polynomial $f$ of degree $k \in \mathbb{N}$. Prove the mean value formula

$$D^\alpha u(x) = ⨍_{B(x,r)} D^\alpha u(y)\,dy \tag{8}$$

for any multi-index $|\alpha| > k$. For what $x \in U$ and $r > 0$ is the formula valid? *(You may use the mean value formula for harmonic functions.)*

**Solution:** If $f$ is a $k$th order polynomial then $D^\alpha f(x) = 0$ for every $|\alpha| > k$. Hence,

$$-\Delta(D^\alpha u) = 0 \qquad \text{in } U$$

for any $|\alpha| > k$, that is, $D^\alpha u$ is harmonic in $U$. Thus, the mean value formula yields (8) for any $x \in U$ and $r > 0$ such that $B(x,r) \subset U$.

### 5d

Use **5c** to prove the following maximum principle for any multi-index $|\alpha| > k$:

$$D^\alpha u(x) \leqslant \max_{\partial U} D^\alpha u \qquad \forall\, x \in U. \tag{9}$$

**Solution:** Let $M = \max_{\overline{U}} D^\alpha u$. Then either $D^\alpha u(x) < M$ for all $x \in U$ (which clearly implies (9)), or there is some $x^0 \in U$ where $D^\alpha u(x^0) = M$. Assume the latter. For any $x \in U$ and $r > 0$ with $B(x,r) \subset U$ we have

$$D^\alpha u(x) = ⨍_{B(x,r)} D^\alpha u(y)\,dy \leqslant ⨍_{B(x,r)} M\,dy = M,$$

with equality *if and only if* $D^\alpha u(y) = M$ for all $y \in B(x,r)$. Setting $x = x^0$ in the above computation yields equality between the left- and right-hand sides, so $D^\alpha u$ must be constant in $B(x,r)$. To show that $D^\alpha u(\bar{x}) = M$ at any other point $\bar{x} \in U$ we select balls $B(x^i, r^i) \subset U$ for $i = 0, \dots, N$ so that $x^i \in B(x^{i-1}, r^{i-1})$ for every $i = 1, \dots, N$ and such that $\bar{x} \in B(x^N, r^N)$. (This is possible since $U$ is connected.) Repeating the above argument reveals that $D^\alpha u \equiv M$ in each ball, and hence also at $\bar{x}$. We conclude that $D^\alpha u \equiv M$ in $U$, and in particular, (9) holds.

## Problem 6 (weight 25%)

~~Let $U \subset \mathbb{R}^n$ be open, bounded and connected.~~ Consider the advection-diffusion problem

$$\begin{cases} u_t + f(u)_x = \varepsilon u_{xx} & \text{for } x \in (0,1),\ t \in (0,T] \\ u(0,t) = u(1,t) = 0 & \text{for } t \in (0,T] \\ u(x,0) = g(x) & \text{for } x \in (0,1) \end{cases} \tag{10}$$

*(Continued on page 8.)*

where $f(u) = u^3$, $\varepsilon > 0$ is a given number and $g \in C([0,1])$ satisfies $g(0) = g(1) = 0$. Let $u \in C^2((0,1) \times (0,T]) \cap C([0,1] \times [0,T])$ be a solution of (10).

## 6a Energy method

Prove that $E[u](t) := \int_0^1 u(x,t)^2\,dx$ decreases over time.

**Solution:**

$$\frac{d}{dt}E[u](t) = \int_0^1 2u(x,t)u_t(x,t)\,dx = 2\int_0^1 \varepsilon u(x,t)u_{xx}(x,t) - u(x,t)f(u(x,t))_x\,dx$$

(integration by parts and the chain rule)

$$= 2\varepsilon \underbrace{u(x,t)u_x(x,t)\Big|_{x=0}^{x=1}}_{=0,\text{ by the BC}} - 2\varepsilon\int_0^1 u_x(x,t)^2\,dx - 3\int_0^1 u(x,t)^3 u(x,t)_x\,dx$$

$$= \underbrace{-2\varepsilon\int_0^1 u_x(x,t)^2\,dx}_{\leqslant 0} - \frac{3}{4}\int_0^1 \left(u(x,t)^4\right)_x\,dx$$

(integration by parts)

$$\leqslant \underbrace{-\frac{3}{4}u(x,t)^4\Big|_{x=0}^{x=1}}_{=0,\text{ by the BC}}.$$

## 6b Maximum principle

Prove that $\min_{y\in[0,1]} g(y) \leqslant u(x,t) \leqslant \max_{y\in[0,1]} g(y)$ for every $x \in [0,1]$, $t \in [0,T]$.

*Hint: Prove the result for $v^\delta(x,t) = u(x,t) - \delta t$ for some $\delta > 0$ first. What equation does $v^\delta$ satisfy?*

**Solution:** Let $v^\delta(x,t) = u(x,t) - \delta t$ for some $\delta > 0$. Then $v^\delta_x = u_x$ and

$$v^\delta_t = u_t - \delta = \varepsilon u_{xx} - f(u)_x - \delta = \varepsilon v^\delta_{xx} - f(v^\delta + \delta t)_x - \delta,$$

so writing $f(v^\delta + \delta t)_x = f'(v^\delta + \delta t)v^\delta_x$ gives

$$v^\delta_t + f'(v^\delta + \delta t)v^\delta_x - v^\delta_{xx} < 0.$$

Assume that $v^\delta$ attains a maximum at some point $(x^0, t^0) \in (0,1) \times (0,T]$. Then

$$v^\delta_x(x^0,t^0) \geqslant 0, \qquad v^\delta_x(x^0,t^0) = 0, \qquad v^\delta_{xx}(x^0,t^0) \leqslant 0,$$

so

$$0 > v^\delta_t(x^0,t^0) + f'(v^\delta(x^0,t^0) + \delta t^0)v^\delta_x(x^0,t^0) - v^\delta_{xx}(x^0,t^0) \geqslant 0,$$

(Continued on page 9.)

a contradiction. Hence, $v^\delta$ attains its maximum somewhere along the set

$$\Gamma = \{(x,t) \ : \ t = 0 \text{ or } x \in \{0,1\}\}.$$

Passing $\delta \to 0$ yields for any $(x,t) \in [0,1] \times [0,T]$

$$u(x,t) = \lim_{\delta\to 0} v^\delta(x,t) \leqslant \lim_{\delta\to 0} \max_\Gamma v^\delta = \max_\Gamma u$$

(*since* $u = 0$ *at* $x = 0$ *and* $x = 1$)

$$= \max\Big(\max_{y\in[0,1]} g(y),\ 0\Big)$$

(*since* $g(0) = 0$)

$$= \max_{y\in[0,1]} g(y).$$

A similar procedure would yield the lower bound.

## 6c Uniqueness

Unlike for the heat equation, we cannot apply the results in **6a** or **6b** to prove uniqueness of the solution of (10). Why not?

**Solution:** The equation is nonlinear, so if $u, v$ are two solutions then $w = u - v$ is not necessarily a solution. This means that the standard approach to proving uniqueness via *a priori* bounds such as the energy bound or maximum principle will not work.

THE END