Problem sheet for week 1
MAT4301

Ulrik Skre Fjordholm

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Vector calculus

1. Compute the gradient Du of the following functions:

(a) $u(x) = \sin(x_1 x_2^2 - x_3)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

(b) $u(x) = |x|^2$ for $x \in \mathbb{R}^n$

(c) $u(x) = |x|$ for $x \in \mathbb{R}^n$

(Here and elsewhere, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ denotes the Euclidean norm of x.)

2. Write out the expression

$$\sum_{|\alpha|=2} \alpha! x^\alpha$$

where $x = (x_1, x_2)$ is some point in $\mathbb{R}^2$.

(Here and elsewhere we use the convention that α denotes a multiindex, so the sum runs over all pairs of nonnegative integers $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ whose sum $|\alpha| = \alpha_1 + \alpha_2$ equals 2.)

3. Compute the partial derivative $D^\alpha u$ for all multiindices α of length $|\alpha| = 1$ and $|\alpha| = 2$, for each of the functions in problem **1**.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a given function, fix a point $x \in \mathbb{R}^2$ and define $g(t) = f(tx)$. Write out $g'(t)$ and $g''(t)$ in terms of partial derivatives of f. Use multiindex notation.

5. Solve problems 3, 4, 5 in Section 1.5 in Evans.

Hint for 1.5.3: Use induction on n, not k. Recall the binomial theorem, $(a+b)^m = \sum_{r=0}^m \binom{m}{r} a^r b^{m-r}$, where $\binom{m}{r} = \frac{m!}{r!(m-r)!}$ are the binomial coefficients.

Hint for 1.5.4: Use induction on n. Recall Leibniz' formula in one dimension: $\partial_{x_n}^k (fg) = \sum_{r=0}^k \binom{k}{r} \partial_{x_n}^r f \partial_{x_n}^{k-r} g$ for functions $f, g \in C^k(\mathbb{R}^n)$.

Hint for 1.5.5: As mentioned in the exercise, define $g(t) = f(tx)$ for $t \in \mathbb{R}$. Write down the kth order Taylor expansion for $g(1)$ (including error term), expanded around $t = 0$. Show that the mth derivative of g can be written as $g^{(m)}(t) = \sum_{|\alpha|=m} \binom{m}{\alpha} x^\alpha D^\alpha f(tx)$. To this end:

- Show first that

$$g^{(m)}(t) = \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n x_{i_1} \cdots x_{i_k} \partial_{x_{i_1}} \cdots \partial_{x_{i_m}} f(tx).$$

- Next, recall the fact that for a multiindex α of length $|\alpha| = m$, the number $\binom{m}{\alpha} = \frac{m!}{\alpha_1!\cdots\alpha_n!}$ is the number of ways to extract m balls of n different colors from a bag, picking α_1 of the first color, α_2 of the second color, and so on. Use this fact to rewrite the above expression for $g^{(m)}(t)$ in multiindex notation.

Integration

6. Compute the integral

$$\int_{B(0,1)} \operatorname{div} \mathbf{F}(x)\, dx$$

where $B(0,1)$ is the unit ball in $\mathbb{R}^3$ and $\mathbf{F}(x) = |x|^2 x$. What do you get when $B(0,1)$ is the unit ball (or *disc*) in $\mathbb{R}^2$?
Hint: Use the divergence theorem (Theorem 1(ii) in §C.2).

7. Use the Gauss–Green theorem (Theorem 1(i) in §C.2) to prove all of the other identities in §C.2.

8. A function $u : \mathbb{R}^n \to \mathbb{R}$ is *locally integrable* if for every bounded set $K \subset \mathbb{R}^n$, the integral

$$\int_K |u(x)|\, dx$$

is finite.

(a) Show that $u(x) := \log|x|$ for $x \in \mathbb{R}^n$ is locally integrable, for any number of dimensions $n \in \mathbb{N}$.

(b) Let $u(x) := |x|^p$ for $x \in \mathbb{R}^n$ and $p \in \mathbb{R}$ a given number. For what values of p is this function locally integrable?

Hint: If u is bounded on K (i.e., $\exists\, C > 0$ such that $|u(x)| \leqslant C$ for all $x \in K$) then u is integrable over K, so it suffices to concentrate on bounded domains K where u is unbounded.

PDEs

9. Find a function $u : \mathbb{R}^3 \to \mathbb{R}$ satisfying the PDE

$$-\Delta u = 1 \qquad \text{in } \mathbb{R}^3.$$

Hint: Try the function $v(x) = |x|^2$ first.

10. Solve the previous problem with $\mathbb{R}^3$ replaced by $\mathbb{R}^n$, for any $n \in \mathbb{N}$.

11. Solve problem 1.5.1 in Evans.