Problem sheet for week 12
MAT4301

Ulrik Skre Fjordholm

Updated November 8, 2019

1. (a) Write down d'Alembert's formula for the solution of the one-dimensional wave equation with wave speed $c > 0$,

$$\begin{cases} u_{tt} = c^2 u_{xx} & (x \in \mathbb{R},\ t > 0) \\ u(x,0) = g(x) \\ u_t(x,0) = h(x). \end{cases} \tag{1}$$

Draw the backwards wave cone from a point $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

(b) Solve (1) when $c = 1$, $g(x) = \sin(x)$, $h(x) = \cos(2x)$.

2. Consider the one-dimensional wave equation on $\mathbb{R}_+$ with *Neumann boundary condition,*

$$\begin{cases} u_{tt} = c^2 u_{xx} & (x > 0,\ t > 0) \\ u(x,0) = g(x) & (x > 0) \\ u_t(x,0) = h(x) & (x > 0) \\ u_x(0,t) = 0 & (t > 0). \end{cases} \tag{2}$$

(a) Find a solution formula for (2).
Hint: See [p. 69, Evans]. Make an even *reflection.*

(b) Make a physical interpretation of the boundary condition and of your solution formula.

3. *(Problem 2.5.23 in Evans)* Let S denote the square lying in $\mathbb{R} \times (0,\infty)$ with corners at the points $(0,1)$, $(1,2)$, $(0,3)$, $(-1,2)$. Define

$$f(x,t) := \begin{cases} -1 & \text{for } (x,t) \in S \cap \{t > x + 2\} \\ 1 & \text{for } (x,t) \in S \cap \{t \leqslant x + 2\} \\ 0 & \text{otherwise.} \end{cases}$$

Assume that u solves

$$\begin{cases} u_{tt} = u_{xx} + f & \text{in } \mathbb{R} \times (0,\infty) \\ u = 0,\ u_t = 0 & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Describe the shape of u for all $(x,t) \in \mathbb{R} \times (0,\infty)$. Draw the graph of $x \mapsto u(x,t)$ for $t > 3$.

4. Use the method of characteristics to solve the following problems:

(a)

$$\begin{cases} u_t + (1-t)u_y = -u & \text{for } y \in \mathbb{R},\ t > 0 \\ u(y,0) = g(y) & \text{for } y \in \mathbb{R} \end{cases}$$

(b)

$$\begin{cases} u_t + y u_y = 0 & \text{for } y > 0,\ t > 0 \\ u(y,0) = \sin(y) & \text{for } y > 0 \\ u(0,t) = t & \text{for } t > 0 \end{cases}$$

Warning: The noncharacteristic condition is not satisfied along the boundary $y = 0$, and therefore the boundary condition $u(0,t) = t$ should be ignored.

(c)

$$\begin{cases} u_t + a u_y = 0 & \text{for } y \in \mathbb{R},\ t > 0 \\ u(y,0) = g(y) & \text{for } y \in \mathbb{R} \end{cases}$$

where $a : \mathbb{R} \to \mathbb{R}$ is the function

$$a(y) = \begin{cases} 0 & \text{for } y < 0 \\ y & \text{for } 0 \leqslant y \leqslant 1 \\ 1 & \text{for } 1 < y. \end{cases}$$