Problem sheet for week 3
MAT4301

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1. *(Problem 2.5.4 in Evans)* Give a direct proof that if $u \in C^2(U) \cap C(\bar{U})$ is harmonic within a bounded open set U, then

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

Hint: Define $u_\varepsilon := u + \varepsilon |x|^2$ for $\varepsilon > 0$, and show u_ε cannot attain its maximum over $\bar{U}$ at an interior point.

Hint: Prove the above by contradiction by assuming that u_ε has a maximum at some $x_0 \in U$. What is then $\Delta u_\varepsilon(x_0)$?

2. *(Problem 2.5.5 in Evans)* We say $v \in C^2(U) \cap C(\bar{U})$ is *subharmonic* if

$$-\Delta v \leqslant 0 \qquad \text{in } U. \tag{1}$$

(a) Prove for subharmonic v that

$$v(x) \leqslant \fint_{B(x,r)} v\, dy \qquad \text{for all } B(x,r) \subset U. \tag{2}$$

(b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.

(c) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that u is harmonic and $v := \varphi(u)$. Prove v is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

3. Repeat problem **1** for *subharmonic* functions.

4. *(Problem 2.5.6 in Evans)* Let U be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_{\bar{U}} |u| \leqslant C\big(\max_{\partial U} |g| + \max_{\bar{U}} |f|\big) \tag{3}$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases} \tag{4}$$

Hint: $-\Delta\big(u + \frac{|x|^2}{2n}\lambda\big) \leqslant 0$ for $\lambda := \max_{\bar{U}} |f|$. Use the same technique as in problem **1**.