Problem sheet for week 5
MAT4301

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1. Let $U, V \subset \mathbb{R}^n$ be sets such that U is open and $V \subset\subset U$ (i.e. $\bar{V}$ is compact and $\bar{V} \subset U$). Prove that $\operatorname{dist}(V, \partial U) > 0$, where

$$\operatorname{dist}(A, B) := \inf_{x\in A, y\in B} |x - y| \qquad \text{for } A, B \subset \mathbb{R}^n.$$

(Thus, "V does not touch the boundary of U".)

2. *(Maximum principle)* Let $U \subset \mathbb{R}^n$ be open and bounded and consider the *advection-diffusion problem*

$$b \cdot Du = \mu \Delta u \qquad (\text{in } U) \tag{1}$$

where $b \in \mathbb{R}^n$ is a fixed vector (the *velocity*) and $\mu > 0$ is a given number (the *diffusion coefficient* or *viscosity*). Prove the maximum principle

$$u(x) \leqslant \max_{y\in\partial U} u(y) \qquad \forall\, x \in U$$

for any function $u \in C^2(U) \cap C(\bar{U})$ satisfying (1).

Hint: Let $v_\varepsilon(x) = u(x) + \varepsilon\big(|x|^2 - Mb \cdot x\big)$ for some $M > 0$. Find an M so that v_ε cannot attain its maximum in U. Now use the technique from problem 1 in problem sheet 3.

3. *(Variational formulation of Laplace's equation)* Consider the Laplace problem

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \tag{2}$$

for some $g \in C(\partial U)$. We will show that a function $u \in C^2(U) \cap C(\bar{U})$ solves (2) *if and only if* it is the (unique) solution of the minimization problem

$$\text{minimize } E(u) := \int_U |Du|^2\, dx \qquad \text{subject to the condition } u = g \text{ on } \partial U \tag{3}$$

(E is the *Dirichlet energy*).

(a) Define the set $D_g := \big\{u \in C^2(U) \cap C(\bar{U}) \;:\; u = g \text{ on } \partial U\big\}$. Show that $E(u)$ is strictly convex on the set D_g.

(b) Show that, as a consequence of (a), the minimizer of $E(u)$ over $u \in D_g$ is unique (if it exists).

(c) Compute the Fréchet derivative of E, that is, find a functional $F(u, \varphi)$ so that

$$E(u + h\varphi) = E(u) + hF(u, \varphi) + O(h^2) \qquad \text{for small } h > 0.$$

(Note in particular that for any $u \in D_g$, we can write all other functions $v \in D_g$ as $v = u + h\varphi$ for some $\varphi \in D_0$, i.e. those $\varphi \in C^2(U) \cap C(\bar{U})$ with $\varphi \equiv 0$ on ∂U.)

(d) Let u be a solution of (2). Show that u minimizes E by showing that $F(u, \varphi) = 0$ for all $\varphi \in D_0$.

(e) Conversely, show that a function $u \in D_g$ which minimizes E satisfies (2).

Remark (Interpretation of problem **3**)**.** Imagine an elastic membrane stretched over a region $U \subset \mathbb{R}^2$ in the x_1-x_2-plane, and whose height above a point $x \in U$ is $u(x)$. Let us prescribe the height profile $g(x) = u(x)$ along the boundary $x \in \partial U$. We ask:

What is the resulting shape of the membrane?

The principle of least action states that the membrane will take the shape which requires the least amount of energy. The (density of the) energy due to stretching is (proportional to) $|Du|^2$, so $E(u)$ is the total energy of the shape prescribed by u. We conclude that the "shape function" u is the solution of the minimization problem (3), which (by problem **3**) is precisely the solution of the Laplace problem (2).

Remark. We call (2) the *Euler–Lagrange equation* corresponding to the minimization problem (3).

4. *(Variational formulation of Poisson's equation)* Find a functional $E : D_g \to \mathbb{R}$ whose (unique) minimizer is the solution of the Poisson problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \tag{4}$$

for $f \in C(\bar{U})$ and $g \in C(\partial U)$.