Problem sheet for week 6
MAT4301

Ulrik Skre Fjordholm

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1. *(The energy method for Poisson's problem)* Let $U \subset \mathbb{R}^n$ be open and bounded and consider the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \tag{1}$$

for $f \in C(\bar{U})$ and $g \in C(\partial U)$. The *Dirichlet energy* of a function $u \in C^1(U)$ is

$$E(u) := \int_U |Du(x)|^2 \, dx. \tag{2}$$

(a) Let $u, v \in C^2(U) \cap C(\bar{U})$ be two solutions of (1). Show that the *relative energy* $E(u-v)$ between u and v is zero.

(b) Conclude that there exists at most one solution of (1).

(c) Let now $u, v \in C^2(U) \cap C(\bar{U})$ be two solutions of (1) but with different data f, g and $\tilde{f}, \tilde{g}$, respectively. Use the technique above to estimate the relative energy $E(u-v)$ in terms of $f - \tilde{f}$ and $g - \tilde{g}$. (This yields a *stability estimate*: If $f, \tilde{f}$ and $g, \tilde{g}$ are close, then so are u, v.)

2. *(The variational principle)*

(a) Let $U \subset \mathbb{R}^n$ be open and let $v \in C(U)$ be such that

$$\int_U v(x)\varphi(x) \, dx = 0 \qquad \forall \, \varphi \in C_c^\infty(U).$$

Show that $v \equiv 0$.
Recall: $C_c^\infty(U) = \{\varphi \in C^\infty(U) \ : \ \operatorname{supp}\varphi \text{ is bounded}\}$, where $\operatorname{supp}\varphi$ (the support of φ) is the closure of the set $\{x \in U : \varphi(x) \neq 0\}$.
Hint: Assume conversely that $v(x^0) \neq 0$ and let φ be a function which is nonzero only very close to x^0.

(b) Let $u \in C^1(U)$ and assume that $v \in C(U)$ is such that

$$\int_U u(x)\varphi_{x_i}(x) \, dx = -\int_U v(x)\varphi(x) \, dx \qquad \forall \, \varphi \in C_c^\infty(U)$$

for some $i \in \{1, \dots, n\}$. Find an expression for v in terms of u.

3. *(Poisson's formula for half-space)* Read carefully through the proof of Theorem 14 in Section 2.2 in Evans. Try to prove (34), at least in the case $n = 2$ (hint: use "polar coordinates" in the plane $\partial\mathbb{R}^n_+$). Try also to prove the claim "$\to 0$ as $x_n \to 0^+$." at the bottom of p. 38.

*Note: We will not go through Section c, Green's function for a ball. Section 2.2.5a was covered in Problem **1**, and Section 2.2.5b was covered in the previous problem sheet.*