Problem sheet for week 8
MAT4301

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1. (a) Find a solution formula for the initial value problem

$$\begin{cases} u_t = k\Delta u & (x \in \mathbb{R}^n, t > 0) \\ u(x,0) = g(x) & (x \in \mathbb{R}^n) \end{cases} \tag{1}$$

for some $k > 0$ and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.
Recall: k is the heat conductivity, indicating how quickly heat is transported through the material.

(b) What would happen for negative k?

(c) Find a solution formula for the initial value problem

$$\begin{cases} u_t(x,t) = k(t)\Delta u(x,t) & (x \in \mathbb{R}^n, t > 0) \\ u(x,0) = g(x) & (x \in \mathbb{R}^n) \end{cases} \tag{2}$$

for a continuous function $k : [0,\infty) \to (0,\infty)$.

2. *(Problem 2.5.12 from Evans)* Suppose u is smooth and solves $u_t = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$.

(a) Show that $u^\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) Use (a) to show that $v(x,t) := x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.
Hint: Compute $\frac{\partial}{\partial\lambda} u^\lambda(x,t)$.

3. *(Problem 2.5.14 from Evans)* Write down an explicit formula for a solution of

$$\begin{cases} u_t + cu = \Delta u + f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{3}$$

where $c \in \mathbb{R}$ is a constant.

Hint: Pretend first that the Δu term is not there and multiply the equation by an integrating factor. Reduce the problem to one for which you already know the solution.

Remark: The term cu is a *reaction term* – it will "produce" or "remove" heat at a rate proportional to $u(x,t)$.

4. *(Part of the proof of Theorem 6)* Let $T, \varepsilon > 0$. Show that the function

$$w(x,t) = \frac{1}{(T+\varepsilon-t)^{n/2}} e^{|x|^2/4(T+\varepsilon-t)}$$

satisfies the heat equation $w_t = \Delta w$ for $t \in (0,T]$, $x \in \mathbb{R}^n$. Note that $w(x,t)$ increases very quickly as $|x| \to \infty$.