Deadline: Thursday 29. October 2020

MANDATORY ASSIGNMENT MAT4301 — FALL 2020

INFORMATION

All mandatory assignments must be uploaded via Canvas.

- *The assignment must be submitted as a single PDF file.*
- *Scanned pages must be clearly legible.*
- *The submission must contain your name, course and assignment number.*

If these requirements are not met, the assignment will not be evaluated. Read the information about mandatory assignments carefully: http://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html

To have a passing grade you must have satisfactory answers to at least 50% of the questions and have seriously attempted to solve all of them.

PROBLEM 1

Let Ω be a connected, bounded and open subset of $\mathbb{R}^n$. A function $u \in C^2(\overline{\Omega})$ is called *superharmonic* (in Ω) if $\Delta u \leq 0$ in Ω.

a)

Suppose u is superharmonic. Prove that

$$u(x) \geq \frac{1}{|B(x,r)|} \int_{B(x,r)} u \, dy,$$

for any ball $B(x,r) \subset \Omega$.

b)

For a superharmonic function u prove that if

$$\min_{\overline{\Omega}} u = u(x_0),$$

for some $x_0 \in \Omega$ (interior minimum), then $u \equiv$ constant.

c)

Suppose $u, v \in C^2(\overline{\Omega})$ satisfy

$$\Delta u \leq 0 \text{ in } \Omega, \quad u = f \text{ on } \partial\Omega \quad \text{(superharmonic)}$$

and

$$\Delta v \geq 0 \text{ in } \Omega, \quad v = g \text{ on } \partial\Omega \quad \text{(subharmonic).}$$

Use **b)** to show that

$$u(x) - v(x) \geq \min_{\partial\Omega}(f - g), \quad x \in \Omega.$$

PROBLEM 2

a)

Suppose $u \in C^2(\mathbb{R}^n \times (0,\infty))$ solves the heat equation. Show that the rescaled function

$$u_\lambda(x, t) = u(\lambda x, \lambda^2 t), \quad \lambda > 0,$$

solves the heat equation as well.

b)

Exploit the result from **a)** to show that the function $v(x, t) = x \cdot Du(x, t) + 2tu_t(x, t)$ solves the heat equation.

c)

Let $S \in C^2(\mathbb{R})$ be a concave function, and assume that u solves the heat equation. Prove that $v := S(u)$ is a supersolution of the heat equation, that is $v_t - \Delta v \geq 0$.

d)

Consider the initial-value (Cauchy) problem

$$\text{(CP-H)} \quad \begin{cases} u_t - \Delta u + cu = f, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where c is a constant, $f \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$, and $u_0 \in C^2(\mathbb{R}^n)$, with f, u_0 compactly supported. Write down a solution formula for (CP-H). Verify that this solution candidate belongs to $C^{2,1}(\mathbb{R}^n \times (0,\infty))$, satisfies the PDE for $x \in \mathbb{R}^n$ and $t > 0$, and satisfies the initial condition in the sense that $u(t,x) \to u_0(x_0)$ as $(x,t) \to (x_0,0)$.

e)

In (CP-H), take $f \equiv 0$ and suppose $u(x,t) \to 0$ as $|x| \to \infty$, for any $t > 0$. Use the energy method to show that the solution u of (CP-H) satisfies

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq e^{-ct}\|u_0\|_{L^2(\mathbb{R}^n)}, \quad t > 0.$$

Use this result to show that there exists *at most one* classical solution of (CP-H), satisfying $u \to 0$ as $|x| \to \infty$.

PROBLEM 3

a)

Consider the function

$$\Phi(x) = \frac{1}{n(n-2)\alpha_n} \frac{1}{|x|^{n-2}}, \qquad x \neq 0,$$

where α_n denotes the volume of the unit ball in $\mathbb{R}^n$ ($n \geq 3$). Verify that

$$\Delta\Phi(x) = 0, \qquad x \neq 0.$$

Moreover, show that

$$|\Phi_{x_i x_i}(x)| \leq \frac{C}{|x|^n}, \qquad x \neq 0, \quad i = 1,\ldots,n\,.$$

b)

Suppose $u \in \mathscr{A} := \left\{w \in C^2(\overline{\Omega}) : w = 0 \text{ on } \partial\Omega\right\}$ is a minimizer of the functional

$$I[w] = \int_\Omega \frac{1}{2}|Dw|^2 - F(w)\,dx, \qquad w \in \mathscr{A},$$

where $F : \mathbb{R} \to \mathbb{R}$ is a C^1 function. Prove that u satisfies the nonlinear Poisson equation

$$-\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

where $f := F'$.