Deadline: Thursday 28. October 2021

MANDATORY ASSIGNMENT MAT4301 (PARTIAL DIFFERENTIAL EQUATIONS)—FALL 2021

INFORMATION

All mandatory assignments must be uploaded via Canvas.

- The assignment must be submitted as a single PDF file.
- Scanned pages must be clearly legible.
- The submission must contain your name, course and assignment number.

If these requirements are not met, the assignment will not be evaluated. Read the information about mandatory assignments carefully:[http://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html].

To have a passing grade you must have satisfactory answers to at least 50% of the questions and have attempted to solve all of them.

PROBLEM 1

a)

Suppose $u \in C^2(\mathbb{R}^n)$ is a solution to $\Delta u = 0$ in $\mathbb{R}^n$ and satisfies

$$|u(x)| \leq K|x|^a, \quad x \in \mathbb{R}^n,$$

for some constants $K > 0$ and $a \in (0,1)$. Show that u must necessarily be constant.

b)

Let Ω be a bounded and open subset of $\mathbb{R}^n$. Consider a second order linear partial differential operator L defined by

$$L[u] = \sum_{i,j=1}^{n} a_{i,j}(x)\partial^2_{x_i x_j} u + \sum_{i=1}^{n} b_i(x) \cdot \partial_{x_i} u + c(x), \quad u \in C^2(\Omega),$$

where $a_{i,j}, b_i, c$ are continuous functions on Ω. Besides, $a_{i,j} = a_{j,i}$ for all i, j, so that the matrix $a = \left\{a_{i,j}\right\}_{i,j=1}^{n}$ is symmetric. We say that L is *uniformly elliptic* if there is a number $\lambda > 0$ such that $\sum_{i,j} a_{i,j}(x)\xi_i\xi_j \geq \lambda |\xi|^2$ for all $\xi \in \mathbb{R}^n$, for all $x \in \Omega$.

(i) Let $\gamma(x)$ be a continuous function that is lower bounded by a number $\gamma_0 > 0$ for all x. Show that $L[u] = \gamma(x)\Delta u$ is uniformly elliptic.

(ii) Suppose $n = 2$, and let $\alpha = \alpha(x), \beta = \beta(x)$ be continuous functions on $\Omega \subset \mathbb{R}^2$. Show that

$$L[u] = \left(1 + \alpha^2(x)\right) \partial^2_{x_1 x_1} u + 2\alpha(x)\beta(x)\partial^2_{x_1 x_2} u + \left(1 + \beta^2(x)\right) \partial^2_{x_2 x_2} u$$

is a uniformly elliptic partial differential operator.

c)

Let Ω be a connected, bounded and open subset of $\mathbb{R}^n$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies the uniformly elliptic PDE

$$L[u] := \Delta u + b \cdot Du = 0 \quad \text{in } \Omega,$$

where $b = b(x) = \left(b_1(x), \ldots, b_n(x)\right)$ is a vector of continuous functions. Establish the weak maximum principle, which asserts that

$$\min_{\partial\Omega} u \leq u(x) \leq \max_{\partial\Omega} u, \qquad x \in \overline{\Omega}.$$

Hint: Show that $u_\varepsilon(x) := u + \varepsilon v(x)$, $\varepsilon > 0$, cannot attain its maximum at an interior point of Ω, where $v(x) = e^{cx_1}$ and $c > 0$ is a suitably chosen constant.

d)

Prove that there exists at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the boundary value problem

$$\Delta u + b \cdot Du = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega,$$

where $b = b(x) = \big(b_1(x), \ldots, b_n(x)\big) : \Omega \to \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ are given continuous functions.

PROBLEM 2

Let Ω be a connected, bounded and open subset of $\mathbb{R}^n$. A function $u \in C^2(\overline{\Omega})$ is called *superharmonic* (in Ω) if $\Delta u \leq 0$ in Ω. Suppose u is superharmonic. Then it is known that

$$u(x) \geq \frac{1}{|B(x,r)|} \int_{B(x,r)} u \, dy,$$

for any ball $B(x,r) \subset \Omega$ (mean value formula for superharmonic functions).

a)

For a superharmonic function u prove that if

$$\min_{\overline{\Omega}} u = u(x_0),$$

for some $x_0 \in \Omega$ (interior minimum), then $u \equiv$ constant.

b)

Suppose $u \in C^2(\overline{\Omega})$ satisfies the (nonlinear) PDE $\Delta u + f(u) = 0$ in Ω and $u = 0$ on $\partial\Omega$, where $f : \mathbb{R} \to \mathbb{R}$ is a given function.

(i) Suppose f is positive, $f \geq 0$. Prove that either $u > 0$ in Ω or $u \equiv 0$.

(ii) Suppose $f(u) = u\left(1 - u^2\right)$. Show that $u(x) \leq 1$ for all $x \in \Omega$. Hint: Argue by contradiction, assuming that $M := \max_{\overline{\Omega}} u > 1$. Similarly, prove that $u \geq -1$.

c)

Let $u \in C^2(\mathbb{R}^n)$ solve the Poisson equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^n, \qquad \text{where } f \in C(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

Suppose $u \to 0$ as $|x| \to \infty$, sufficiently fast to justify integration by parts. Show that

(2c—1) $$\left\| u_{x_i x_j} \right\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}, \qquad i, j = 1, \ldots, n.$$

Remark: From the Poisson equation, it follows immediately that the sum of the second order "diagonal" partial derivatives $u_{x_i x_i}$ is a square-integrable function, if the right-hand side f is. The result (2c—1) is surprisingly much stronger; it shows that each individual second order partial derivative u_{x_i, x_j} (not only the diagonal ones!) is a square-integrable function.

PROBLEM 3

a)

Let Ω be a bounded open set in $\mathbb{R}^n$. Suppose that $u \in C^{2,1}(\Omega \times (0,\infty))$ satisfies

the parabolic PDE

$$u_t - \Delta u + c(x,t)u = 0 \quad \text{in } \Omega \times (0,\infty),$$

where $c = c(x,t)$ is a continuous function, $M := \|c\|_{L^\infty} < \infty$. Let $S \in C^2(\mathbb{R})$ be a positive convex function satisfying

$$\left| zS'(z) \right| \leq KS(z), \quad z \in \mathbb{R},$$

for some constant $K > 0$. Show that $S(u)$ satisfies the inequality

(3a—1) $$S(u)_t - \Delta S(u) \leq CS(u) \quad \text{in } \Omega \times (0,\infty),$$

where $C = MK > 0$ is a constant.

b)

Let Ω be a bounded open set in $\mathbb{R}^n$, with outward unit normal vector ν. Consider the problem

(3b–1) $$\begin{aligned} &u_t - \Delta u + c(x,t)u = f, \quad x \in \Omega,\ t > 0, \\ &u(x,0) = u_0(x), \quad x \in \Omega, \\ &u(x,t) = h(x,t), \quad x \in \partial\Omega,\ t \geq 0, \end{aligned}$$

where c, u_0, h are continuous functions. Prove that there exists a constant $L > 0$ such that

(3b–2) $$\left\| u(\cdot,t) - v(\cdot,t) \right\|_{L^2(\Omega)} \leq e^{Lt} \left\| u_0 - v_0 \right\|_{L^2(\Omega)},$$

where u, v are two classical solutions of (3b—1) with initial functions u_0, v_0, respectively. Explain why this stability result immediately implies the uniqueness of solutions.

c)

Let $\Phi(x,t) = \dfrac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ be the fundamental solution of the heat equation. Let $f(x,t)$ be a compactly supported continuous function. Prove that

$$\lim_{s \downarrow 0} \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\, dy = f(x,t).$$