# UNIVERSITY OF OSLO

## Faculty of mathematics and natural sciences

Exam in: MAT4301 — Partial differential equations

Day of examination: 18 December 2023

Examination hours: 09:00 – 13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

**Note:**

- You can get a total of 100 points. The point distribution is specified for each problem.
- All answers must be justified.

## Problem 1 (25p)

Consider the heat equation

$$\begin{cases} u_t = \Delta u & \text{for } x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n \end{cases} \tag{1}$$

for some continuous, bounded function $g\colon \mathbb{R}^n \to \mathbb{R}$.

### 1a (15p)

Write down the formula for the solution of (1), and use this formula to show that

$$\inf_{\mathbb{R}^n} g \leqslant u(x,t) \leqslant \sup_{\mathbb{R}^n} g \qquad \text{for all } x \in \mathbb{R}^n,\ t > 0.$$

### 1b (10p)

How "smooth" (or "regular") can we expect the above solution $u$ to be? Justify your claim.

## Problem 2 (50p)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected. A function $u \in C^2(\Omega)$ is a *subsolution* of Laplace's equation if $\Delta u(x) \geqslant 0$ for all $x \in \Omega$.

*(Continued on page 2.)*

### 2a (10p)

Find an example of $\Omega \subset \mathbb{R}^3$ and two *different* subsolutions $u, v$ of Laplace's equation on $\Omega$ satisfying $u = v = 0$ on $\partial\Omega$. You can for instance let $\Omega = B(0,1)$.

### 2b (15p)

Let $u$ be a subsolution of Laplace's equation on some domain $\Omega$. For a fixed $x^0 \in \Omega$, define $\varphi(r) := \fint_{B(x^0,r)} u(y)\,dy$ (for $r > 0$ small enough so that $B(x^0, r) \subset \Omega$). Show that $\varphi'(r) \geqslant 0$ for all $r$.

*Hint: You might need to use both change of variables, polar coordinates, and the divergence theorem.*

### 2c (10p)

Use Problem 2b to prove the strong maximum principle: If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a subsolution of Laplace's equation, and there is a point $x^0 \in \Omega$ where $u(x^0) = M := \max_{\overline{\Omega}} u$, then $u$ is constant.

### 2d (15p)

Let $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ be solutions to

$$\begin{cases} \Delta u_1 = f_1 & \text{in } \Omega \\ u_1 = g_1 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta u_2 = f_2 & \text{in } \Omega \\ u_2 = g_2 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $f_1, f_2 \in C(\Omega)$ and $g_1, g_2 \in C(\partial\Omega)$ satisfy $f_1 \leqslant f_2$ and $g_1 \geqslant g_2$. Use Problem 2c to show that $u_1 \geqslant u_2$.

## Problem 3 (15p)

Consider the conservation law

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = g(x). \end{cases} \tag{3}$$

Use the method of characteristics to find a weak solution of (3) when $f(u) = u^2$ and

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x \leqslant 1 \\ 1 & \text{if } 1 < x. \end{cases}$$

Sketch the characteristic curves in the $x$-$t$-plane.

## Problem 4 (10p)

Consider the Hamilton–Jacobi equation

$$\begin{cases} u_t + \dfrac{(u_x)^2}{2} = 0 & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = |x| & \text{for } x \in \mathbb{R}. \end{cases} \tag{4}$$

*(Continued on page 3.)*

Prove that this problem satisfies the conditions necessary to apply the Hopf–Lax formula. Then use this formula to deduce the following expression for the solution:

$$u(x,t) = \begin{cases} |x| - \frac{t}{2} & \text{if } |x| > t \\ \frac{x^2}{2t} & \text{if } |x| \leqslant t. \end{cases} \tag{5}$$

**THE END**