October 31, 2023

MAT4301 – Partial differential equations

Mandatory assignment 1 of 1

Submission deadline

Thursday 16 November 2023 at 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between writing in English or Norwegian.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF. Scanned pages must be clearly legible; please use either the "Color" or "Grayscale" settings. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. As a rule of thumb, you need to correctly answer 2/3 of the assignment in order to get it passed, and to show that you have made an attempt at solving all of the problems.

All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Preliminaries

We first mention some results you might need:

Definition. *Let $n, m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$. We define the L^2-norm of a function $v\colon \Omega \to \mathbb{R}^m$ as the number*

$$\|v\|_{L^2(\Omega)} = \left(\int_\Omega |v(x)|^2\,dx\right)^{1/2}.$$

The set $L^2(\Omega, \mathbb{R}^m)$ is the set of all functions from Ω to $\mathbb{R}^m$ whose L^2-norm is finite. When $m = 1$ we write $L^2(\Omega) = L^2(\Omega, \mathbb{R})$.

Theorem (Poincaré's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then there exists a constant $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leqslant C\,\|Du\|_{L^2(\Omega)} \tag{1}$$

for every $u \in C^1(\Omega) \cap C(\overline{\Omega})$ satisfying $u\big|_{\partial\Omega} \equiv 0$.

Theorem (Gronwall's inequality)**.** *If $\alpha \in C^1([0,\infty))$ satisfies*

$$\alpha'(t) \leqslant b(t)\alpha(t) \qquad \forall\, t > 0 \tag{2}$$

for some $b \in C([0,\infty))$ then

$$\alpha(t) \leqslant \alpha(0)e^{B(t)} \qquad \forall\, t \geqslant 0, \tag{3}$$

where $B(t) = \int_0^t b(s)\,ds$.

Problems

1. *(Long-term behavior of the heat equation)* Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain. Consider the initial-boundary value problem

$$\begin{cases} u_t = \Delta u + f & \text{in } \Omega \times (0,\infty) \\ u = g & \text{on } \partial\Omega \times [0,\infty) \\ u = h & \text{on } \Omega \times \{t = 0\} \end{cases} \tag{4}$$

where the source term f, boundary data g and initial data h are given and $f \in C(\overline{\Omega} \times [0,\infty))$, $g \in C(\partial\Omega \times [0,\infty))$ and $h \in C(\overline{\Omega})$ (where $\overline{\Omega}$ is the closure of Ω).

(a) Let $u, v \in C_1^2(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$ both satisfy (4) with the same functions f and g, but with different initial data h_1 and h_2, respectively. Prove the *energy estimate*

$$\frac{d}{dt}E[u - v](t) = -\int_\Omega |D(u - v)(x,t)|^2\,dx \qquad \forall\, t > 0 \tag{5}$$

where $E[w](t) = \frac{1}{2}\int_\Omega |w(x,t)|^2\,dx$.

(b) Prove that there is a constant $C > 0$ such that

$$E[u - v](t) \leqslant \frac{1}{2}\|h_1 - h_2\|^2_{L^2(\Omega)} e^{-Ct} \qquad \forall\, t \geqslant 0. \tag{6}$$

Hint. *Use* (5) *and the Poincaré and Gronwall inequalities.*

(c) Conclude that the difference between u and v (in an appropriate norm) converges to 0 as $t \to \infty$, no matter how h_1, h_2 were chosen.

Remark. *We can not conclude from the above that u and v converge to some common limit as $t \to \infty$, only that they "become more and more similar".*

(d) Assume now that f and g are constant in time, i.e. $f(x,t) = \bar{f}(x)$ for all $x \in \overline{\Omega}$, $t > 0$ and $g(x,t) = \bar{g}(x)$ for all $x \in \partial\Omega$, $t > 0$, where $\bar{f} \in C(\overline{\Omega})$ and $\bar{g} \in C(\partial\Omega)$. Let u solve (4) as before, and let $\bar{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ solve the Poisson problem

$$\begin{cases} -\Delta \bar{u} = \bar{f} & \text{in } \Omega \\ \bar{u} = \bar{g} & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Show that $u(\cdot, t) \to \bar{u}$ in $L^2(\Omega)$ as $t \to \infty$. How fast does this convergence take place?

2. *(Maximum principles for the wave equation)* Consider the wave equation

$$\begin{cases} u_{tt} = c^2 \Delta u & \text{for } x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n \\ u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}^n \end{cases} \tag{8}$$

for some constant $c > 0$ and given functions $g, h\colon \mathbb{R}^n \to \mathbb{R}$.

(a) Write down solution formulas for (8) in $n = 1, 2$ and 3 dimensions.

(b) We know that the (homogeneous) *heat equation* satisfies the maximum principle

$$u(x,t) \leqslant \max_{y \in \mathbb{R}^n} u(y,0) \qquad \forall\, x \in \mathbb{R}^n,\ t > 0 \tag{9}$$

(and likewise for a lower bound). In this problem you may assume $n = 1$.

(i) Show that we should *not* expect that (9) holds for general initial data for the wave equation.

(ii) Give a sufficient condition on the initial data that ensures that (9) holds for the wave equation.

(c) From Problem (b) we know that a maximum principle of the form

$$u \leqslant \max_{\mathbb{R}^n} g$$

is not true in general, but there *is* a maximum principle that also takes into account the values of h. In the cases $n = 1, 3$, state and prove an estimate on the maximum value that $u(x,t)$ can take. *(This is also possible for $n = 2$, but it requires a longer computation.)*

3. *(The method of characteristics)* Use the method of characteristics to solve the following problems (or, if it's not possible, explain why). In the problems where the domain Ω is not given, you should choose a domain Ω where you can uniquely determine the solution.

(a)

$$\begin{cases} x_1 u_{x_1} + x_2 u_{x_2} = 2u, & \\ u(x_1, 1) = g(x_1) & \text{for } x_1 \in \mathbb{R} \end{cases}$$

(b)

$$\begin{cases} u_t + (1-t)u_y = -u & \text{for } y \in \mathbb{R},\ t > 0 \\ u(y, 0) = g(y) & \text{for } y \in \mathbb{R} \end{cases}$$

(c)

$$\begin{cases} u_t + a u_y = 0 & \text{for } y \in \mathbb{R},\ t > 0 \\ u(y, 0) = g(y) & \text{for } y \in \mathbb{R} \end{cases}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is the function

$$a(y) = \begin{cases} 0 & \text{for } y < 0 \\ y & \text{for } 0 \leqslant y \leqslant 1 \\ 1 & \text{for } 1 < y. \end{cases}$$

(d)

$$\begin{cases} x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u & \\ u(x_1, x_2, 0) = g(x_1, x_2) & \text{for } x_1, x_2 \in \mathbb{R} \end{cases}$$

(e)

$$\begin{cases} u u_{x_1} + u_{x_2} = 1 & \\ u(x_1, x_1) = \frac{1}{2} x_1 & \text{for } x_1 \in \mathbb{R} \end{cases}$$

Hint. *In all of the above problems, first determine F, compute $D_p F$, F_z and $D_x F$, and write down the characteristic equations*

$$\begin{cases} \dot{x} = D_p F(p, z, x) \\ \dot{z} = p \cdot D_p F(p, z, x) \\ \dot{p} = -p F_z(p, z, x) - D_p(p, z, x) \\ F(p, z, x) = 0. \end{cases}$$

Then solve these ODEs (usually just the equations for x, z, if possible) in terms of the initial data (p^0, z^0, x^0). Finally, given an arbitrary point x, find $x^0 \in \Gamma$ and $s \in \mathbb{R}$ such that $x(s) = x$; then $u(x) = u(x(s)) = z(s)$. Since the method of characteristics only provides a candidate solution, *you should check that your function actually solves the problem.*

Hint. *It's always a good idea to draw some of the characteristic curves.*