November 23, 2023

MAT4301 – Partial differential equations

Mandatory assignment 1 of 1

Submission deadline

Thursday 16 November 2023 at 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between writing in English or Norwegian.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF. Scanned pages must be clearly legible; please use either the "Color" or "Grayscale" settings. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. As a rule of thumb, you need to correctly answer 2/3 of the assignment in order to get it passed, and to show that you have made an attempt at solving all of the problems.

All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Preliminaries

We first mention some results you might need:

Definition. *Let $n, m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$. We define the L^2-norm of a function $v\colon \Omega \to \mathbb{R}^m$ as the number*

$$\|v\|_{L^2(\Omega)} = \left(\int_\Omega |v(x)|^2\, dx\right)^{1/2}.$$

The set $L^2(\Omega, \mathbb{R}^m)$ is the set of all functions from Ω to $\mathbb{R}^m$ whose L^2-norm is finite. When $m = 1$ we write $L^2(\Omega) = L^2(\Omega, \mathbb{R})$.

Theorem (Poincaré's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then there exists a constant $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leqslant C\,\|Du\|_{L^2(\Omega)} \tag{1}$$

for every $u \in C^1(\Omega) \cap C(\overline{\Omega})$ satisfying $u\big|_{\partial\Omega} \equiv 0$.

Theorem (Gronwall's inequality)**.** *If $\alpha \in C^1([0,\infty))$ satisfies*

$$\alpha'(t) \leqslant b(t)\alpha(t) \qquad \forall\, t > 0 \tag{2}$$

for some $b \in C([0,\infty))$ then

$$\alpha(t) \leqslant \alpha(0)e^{B(t)} \qquad \forall\, t \geqslant 0, \tag{3}$$

where $B(t) = \int_0^t b(s)\, ds$.

Problems

1. *(Long-term behavior of the heat equation)* Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain. Consider the initial-boundary value problem

$$\begin{cases} u_t = \Delta u + f & \text{in } \Omega \times (0,\infty) \\ u = g & \text{on } \partial\Omega \times [0,\infty) \\ u = h & \text{on } \Omega \times \{t = 0\} \end{cases} \tag{4}$$

where the source term f, boundary data g and initial data h are given and $f \in C(\overline{\Omega} \times [0,\infty))$, $g \in C(\partial\Omega \times [0,\infty))$ and $h \in C(\overline{\Omega})$ (where $\overline{\Omega}$ is the closure of Ω).

(a) Let $u, v \in C_1^2(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$ both satisfy (4) with the same functions f and g, but with different initial data h_1 and h_2, respectively. Prove the *energy estimate*

$$\frac{d}{dt} E[u - v](t) = -\int_\Omega |D(u - v)(x,t)|^2\, dx \qquad \forall\, t > 0 \tag{5}$$

where $E[w](t) = \frac{1}{2}\int_\Omega |w(x,t)|^2\, dx$.

(b) Prove that there is a constant $C > 0$ such that

$$E[u-v](t) \leqslant \frac{1}{2}\|h_1 - h_2\|^2_{L^2(\Omega)} e^{-Ct} \qquad \forall\, t \geqslant 0. \tag{6}$$

Hint. *Use* (5) *and the Poincaré and Gronwall inequalities.*

(c) Conclude that the difference between u and v (in an appropriate norm) converges to 0 as $t \to \infty$, no matter how h_1, h_2 were chosen.

Remark. *We can not conclude from the above that u and v converge to some common limit as $t \to \infty$, only that they "become more and more similar".*

(d) Assume now that f and g are constant in time, i.e. $f(x,t) = \bar{f}(x)$ for all $x \in \overline{\Omega}$, $t > 0$ and $g(x,t) = \bar{g}(x)$ for all $x \in \partial\Omega$, $t > 0$, where $\bar{f} \in C(\overline{\Omega})$ and $\bar{g} \in C(\partial\Omega)$. Let u solve (4) as before, and let $\bar{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ solve the Poisson problem

$$\begin{cases} -\Delta \bar{u} = \bar{f} & \text{in } \Omega \\ \bar{u} = \bar{g} & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Show that $u(\cdot, t) \to \bar{u}$ in $L^2(\Omega)$ as $t \to \infty$. How fast does this convergence take place?

Solution:

(a) Let $w = u - v$; then w solves (4) with data $f = 0$, $g = 0$, and $h = h_1 - h_2$. We get

$$\begin{aligned} \frac{d}{dt} E[w](t) &= \int_\Omega w w_t \, dx = \int_\Omega w \Delta w \, dx \\ &= \int_{\partial\Omega} w Dw \cdot \nu \, dS(x) - \int_\Omega Dw \cdot Dw \, dx. \end{aligned}$$

The fact that $w = 0$ on $\partial\Omega$ now implies that the first term vanishes, leading to (5).

(b) By Poincaré's inequality, there is some $C > 0$ such that

$$\int_\Omega |w(x)|^2 \, dx \leqslant C \int_\Omega |Dw(x)|^2 \, dx$$

for all $w \in C^1(\Omega) \cap C(\overline{\Omega})$ satisfying $w = 0$ on $\partial\Omega$. Hence, the above also holds for $w(x) = (u-v)(x,t)$ for any $t \geqslant 0$. We conclude from (5) that

$$\frac{d}{dt} E[u-v](t) \leqslant -\frac{1}{C} \int_\Omega |(u-v)(x,t)|^2 \, dx = -\frac{2}{C} E[u-v](t).$$

Now let $\alpha(t) = E[u-v](t)$, so that $\alpha'(t) \leqslant -\frac{2}{C}\alpha(t)$. From Gronwall we get that

$$\alpha(t) \leqslant \alpha(0) e^{-2t/C} \qquad \text{for all } t \geqslant 0.$$

Observing that $\alpha(0) = E[u-v](0) = \frac{1}{2}\|h_1 - h_2\|^2_{L^2(\Omega)}$, we arrive at (6).

(c) From (b) we can conclude that $(u - v)(\cdot, t) \to 0$ as $t \to \infty$ in $L^2(\Omega)$.

(d) Let $v(x,t) = \bar{u}(x)$. Then v solves (4) with $h = \bar{u}$, so by (c) we find that

$$E[u - v](t) = \frac{1}{2}\|u(\cdot,t) - \bar{u}\|_{L^2}^2 \leqslant \frac{1}{2}\|h_1 - \bar{u}\|_{L^2}^2 e^{-Ct}.$$

Hence, the L^2 difference between $u(\cdot,t)$ and $\bar{u}$ goes to zero *exponentially fast* as $t \to \infty$.

2. *(Maximum principles for the wave equation)* Consider the wave equation

$$\begin{cases} u_{tt} = c^2 \Delta u & \text{for } x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n \\ u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}^n \end{cases} \tag{8}$$

for some constant $c > 0$ and given functions $g, h \colon \mathbb{R}^n \to \mathbb{R}$.

(a) Write down solution formulas for (8) in $n = 1, 2$ and 3 dimensions.

(b) We know that the (homogeneous) *heat equation* satisfies the maximum principle

$$u(x,t) \leqslant \max_{y \in \mathbb{R}^n} u(y,0) \qquad \forall\, x \in \mathbb{R}^n,\ t > 0 \tag{9}$$

(and likewise for a lower bound). In this problem you may assume $n = 1$.

(i) Show that we should *not* expect that (9) holds for general initial data for the wave equation.

(ii) Give a sufficient condition on the initial data that ensures that (9) holds for the wave equation.

(c) From Problem (b) we know that a maximum principle of the form

$$u \leqslant \max_{\mathbb{R}^n} g$$

is not true in general, but there *is* a maximum principle that also takes into account the values of h. In the cases $n = 1, 3$, state and prove an estimate on the maximum value that $u(x,t)$ can take. *(This is also possible for $n = 2$, but it requires a longer computation.)*

Solution:

(a) We note that if $v(x,t) = u(x, t/c)$ then $v_{tt} = \Delta v$, $v(x,0) = g(x)$ and $v_t(x,0) = h(x)/c$. We can therefore use the solution formulae that we know for this "standard" wave equation:

$n = 1$**:**

$$u(x,t) = v(x,ct) = \frac{1}{2}\big(g(x-ct) + g(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy.$$

$n = 2$**:**

$$u(x,t) = \frac{1}{2}⨍_{B(x,ct)} \frac{ctg(y) + ct^2 h(y) + ctDg(y)\cdot(y - x)}{(c^2t^2 - |y - x|^2)^{1/2}}\,dy.$$

$n = 3$:

$$u(x,t) = \fint_{\partial B(x,ct)} th(y) + g(y) + Dg(y) \cdot (y - x)\, dS(y).$$

(b) Let us assume $n = 1$. Let $g \equiv 1$ and let h be any positive function. Then $u(x,0) \equiv 1$, but the solution is

$$u(x,t) = 1 + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\, dy > 1.$$

Hence, (9) is violated. If, on the other hand, $h \equiv 0$ then

$$u(x,t) = \frac{1}{2}\big(g(x - ct) + g(x + ct)\big) \leqslant \max_y g(y) = \max_y u(y,0).$$

(c) Let us write $\|g\|_\infty = \max_{y \in \mathbb{R}^n} |g(y)|$. Using the solution formulas, we can deduce the following upper bounds:

$n = 1$:

$$\begin{aligned} u(x,t) &\leqslant \|g\|_\infty + \frac{1}{2c} \int_{x-ct}^{x+ct} \|h\|_\infty\, dz \\ &= \|g\|_\infty + t\|h\|_\infty. \end{aligned}$$

$n = 3$: We estimate

$$u(x,t) \leqslant t\|h\|_\infty + \|g\|_\infty + ct\|Dg\|_\infty,$$

where we have used the fact that $|y - x| = ct$ for $y \in \partial B(x,ct)$.

It should be noted that all of the above estimates can be sharpened; for instance, in the case $n = 1$ we could replace $\|h\|_\infty$ by $\max_{y \in [x-ct, x+ct]} h(y)$.

3. *(The method of characteristics)* Use the method of characteristics to solve the following problems (or, if it's not possible, explain why). In the problems where the domain Ω is not given, you should choose a domain Ω where you can uniquely determine the solution.

(a)

$$\begin{cases} x_1 u_{x_1} + x_2 u_{x_2} = 2u, \\ u(x_1, 1) = g(x_1) & \text{for } x_1 \in \mathbb{R} \end{cases}$$

(b)

$$\begin{cases} u_t + (1-t)u_y = -u & \text{for } y \in \mathbb{R},\ t > 0 \\ u(y,0) = g(y) & \text{for } y \in \mathbb{R} \end{cases}$$

(c)

$$\begin{cases} u_t + a u_y = 0 & \text{for } y \in \mathbb{R},\ t > 0 \\ u(y,0) = g(y) & \text{for } y \in \mathbb{R} \end{cases}$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is the function

$$a(y) = \begin{cases} 0 & \text{for } y < 0 \\ y & \text{for } 0 \leqslant y \leqslant 1 \\ 1 & \text{for } 1 < y. \end{cases}$$

(d)

$$\begin{cases} x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u \\ u(x_1, x_2, 0) = g(x_1, x_2) & \text{for } x_1, x_2 \in \mathbb{R} \end{cases}$$

(e)

$$\begin{cases} u u_{x_1} + u_{x_2} = 1 \\ u(x_1, x_1) = \frac{1}{2} x_1 & \text{for } x_1 \in \mathbb{R} \end{cases}$$

Hint. *In all of the above problems, first determine F, compute $D_p F$, F_z and $D_x F$, and write down the characteristic equations*

$$\begin{cases} \dot{x} = D_p F(p, z, x) \\ \dot{z} = p \cdot D_p F(p, z, x) \\ \dot{p} = -p F_z(p, z, x) - D_p(p, z, x) \\ F(p, z, x) = 0. \end{cases}$$

Then solve these ODEs (usually just the equations for x, z, if possible) in terms of the initial data (p^0, z^0, x^0). Finally, given an arbitrary point x, find $x^0 \in \Gamma$ and $s \in \mathbb{R}$ such that $x(s) = x$; then $u(x) = u(x(s)) = z(s)$. Since the method of characteristics only provides a candidate solution*, you should check that your function actually solves the problem.*

Hint. *It's always a good idea to draw some of the characteristic curves.*

Solution:

(a) We have $F(p, z, x) = x_1 p_1 + x_2 p_2 - 2z = x \cdot p - 2z$, so

$$\begin{aligned} D_p F(p, z, x) &= x, \\ F_z(p, z, x) &= -2, \\ D_x F(p, z, x) &= p. \end{aligned}$$

Hence, the characteristic equations are

$$\dot{x} = x, \qquad \dot{z} = x \cdot p = 2z, \qquad \dot{p} = -2p - x = -3p.$$

Hence, $x(s) = x^0 e^s$, $z(s) = z^0 e^{2s}$ and $p(s) = p^0 e^{-3s}$. We have $x^0 \in \Gamma$ if and only if $x_2^0 = 1$, whence $x_2(s) > 0$ for all $s \in \mathbb{R}$. Hence, we can only solve the equation for $x \in \mathbb{R} \times \mathbb{R}_+$. In order for $\Gamma \subset \partial\Omega$, let us set $\Omega := \mathbb{R} \times (1, \infty)$. For an arbitrary $x \in U$ we then have $x(s) = x$ if and only if $e^s = x_2$ and $x_1^0 e^s = x_1$, whence $s = \log x_2$ and $x_1^0 = x_1/x_2$. Therefore,

$$u(x) = u(x(s)) = z(s) = z^0 e^{2s} = g(x_1^0)(e^s)^2 = \underline{\underline{g(x_1/x_2)(x_2)^2}}.$$

A quick check will verify that this function satisfies the given problem. (We note also that the solution formula makes sense, and satisfies the PDE, for all $x_2 > 0$.)

(b) We have $x = (y, t)$, $\Omega = \mathbb{R} \times \mathbb{R}_+$, $\Gamma = \mathbb{R} \times \{0\}$, and $F(p, z, x) = p_2 + (1 - x_2)p_1 + z$. We have

$$\begin{aligned} D_p F(p, z, x) &= \begin{pmatrix} 1 - x_2 \\ 1 \end{pmatrix}, \\ F_z(p, z, x) &= 1, \\ D_x F(p, z, x) &= \begin{pmatrix} 0 \\ -p_1 \end{pmatrix}. \end{aligned}$$

Thus, the characteristic equation is $F(p, z, x) = 0$ and

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 1 - x_2 \\ 1 \end{pmatrix}, \\ \dot{z} &= p_1(1 - x_2) + p_2 = F(p, z, x) - z = -z, \\ \dot{p} &= -p - \begin{pmatrix} 0 \\ -p_1 \end{pmatrix} = \begin{pmatrix} -p_1 \\ -p_1 - p_2 \end{pmatrix}. \end{aligned}$$

The solution for z is clearly $z(s) = z^0 e^{-s}$. For x we have $\dot{t} = 1$ so $t(s) = t_0 + s$, and since $t = 0$ on Γ we get $t_0 = 0$, whence $t(s) = s$. Last, $\dot{y} = 1 - s$, so $y(s) = s - s^2/2 + y_0$. Hence,

$$z(s) = u(y(s), t(s)) = u(s - s^2/2 + y_0, s) = z^0 e^{-s} = g(y^0)e^{-s}.$$

In order to find the solution at an arbitrary point (y, t), set $t = s$ and $y = y(s) = s - s^2/2 + y_0$, so $y_0 = y - s + s^2/2$. Then

$$u(y, t) = \underline{\underline{g\big(y - t + t^2/2\big)e^{-t}}}.$$

(Note that we did not need the equation for p in order to find u.)

(c) Here we have

$$\begin{aligned} D_p F(p, z, x) &= \begin{pmatrix} a(y) \\ 1 \end{pmatrix}, \\ F_z(p, z, x) &= 0, \\ D_x F(p, z, x) &= \begin{pmatrix} a'(y)p_1 \\ 0 \end{pmatrix} \end{aligned}$$

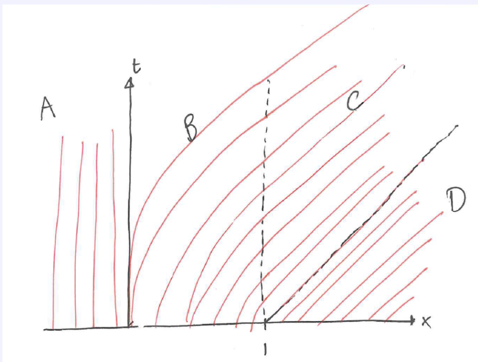
so the characteristic equations are now

$$\begin{aligned} \dot{x} &= \begin{pmatrix} a(y) \\ 1 \end{pmatrix}, \\ \dot{z} &= p_2 + a(y)p_1 = F(p, z, x) = 0, \\ \dot{p} &= -\begin{pmatrix} a'(y)p_1 \\ 0 \end{pmatrix}. \end{aligned}$$

We get $t(s) = s$ as before, and $z \equiv z^0$. We need to solve the equation $\dot{y} = a(y)$: If $y^0 \leqslant 0$ then $y \equiv y^0$. If $0 < y^0 < 1$ then $\dot{y} = y$ for small s, so $y(s) = y^0 e^s$, until the time s^* when $y(s^*) = 1$, at which point $\dot{y} = 1$, whence $y(s) = 1 + s - s^*$. If $y^0 \geqslant 1$ then $y(s) = y^0 + s$. Solving for s^*, we get $s^* = -\log y^0$, and therefore

$$y(s) = \begin{cases} y^0 & \text{if } y^0 \leqslant 0, \\ y^0 e^s & \text{if } 0 < y^0 < 1 \text{ and } s < -\log y^0 \\ 1 + s + \log y^0 & \text{if } 0 < y^0 < 1 \text{ and } s \geqslant -\log y^0 \\ y^0 + s & \text{if } y^0 \geqslant 1. \end{cases}$$

The four domains (labelled A, B, C, D) in the above formula are pictured below:



We know then that

$$u(y(s), s) = z(s) = z^0 = g(y^0).$$

Thus, to find the solution at an arbitrary $(y, s) \in \Omega$, we need to find the corresponding point y^0. If $(y, s) \in A$, i.e. $y \leqslant 0$, then $y^0 = y$. If $(y, s) \in B$, i.e. $0 < y < 1$, then $y = y(s) = y^0 e^s$, so $y^0 = ye^{-s}$. If $(y, s) \in C$, i.e. $y \geqslant 1$ and $s \geqslant y$, then $y = y(s) = 1 + s + \log y^0$, so $y^0 = e^{y-1-s}$. Last, if $(y, s) \in D$, i.e. $y > 1 + s$, then $y = y(s) = y^0 + s$, so $y^0 = y - s$. Putting this together, we find

$$u(y, t) = \begin{cases} g(y) & \text{if } y \leqslant 0 \\ g(ye^{-s}) & \text{if } 0 < y < 1 \\ g(e^{y-1-s}) & \text{if } 1 \leqslant y \leqslant s \\ g(y - s) & \text{if } y > 1 + s. \end{cases}$$

Again, note that we did not use the equation for p.

(d) We have $\Gamma = \mathbb{R} \times \mathbb{R} \times \{0\}$ and $F(p, z, x) = x_1 p_1 + 2x_2 p_2 + p_3 - 3z$, so

$$\begin{aligned} D_p F(p,z,x) &= \begin{pmatrix} x_1 \\ 2x_2 \\ 1 \end{pmatrix}, \\ F_z(p,z,x) &= -3, \\ D_x F(p,z,x) &= \begin{pmatrix} p_1 \\ 2p_2 \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, the characteristic equations are

$$\begin{aligned} \dot{x} &= \begin{pmatrix} x_1 \\ 2x_2 \\ 1 \end{pmatrix}, \\ \dot{z} &= p_1 x_1 + 2p_2 x_2 + p_3 = 3z, \\ \dot{p} &= \begin{pmatrix} 3x_1 - p_1 \\ 6x_2 - 2p_2 \\ 3 \end{pmatrix}. \end{aligned}$$

The solutions to the first two equations are $x_1(s) = x_1^0 e^s$, $x_2(s) = x_2^0 e^{2s}$, $x_3(s) = x_3^0 + s$, $z(s) = z^0 e^{3s}$. Since $x^0 \in \Gamma$ if and only if $x_3^0 = 0$ we get $x_3(s) = s$. Consequently, we can take either $\Omega = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$ or $\Omega = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_-$. Let us choose the former, i.e. $s \geqslant 0$. If we choose an arbitrary $x \in \Omega$ then $s = x_3$, $x_1 = x_1^0 e^s$ so $x_1^0 = e^{-x_3} x_1$, and $x_2 = x_2^0 e^{2s}$, so $x_2^0 = x_2 e^{-2x_3}$. We conclude that

$$u(x) = z(s) = z^0 e^{3s} = g\big(x_1^0, x_2^0\big) e^{3s} = \underline{\underline{g\big(x_1 e^{-x_3}, x_2 e^{-2x_3}\big) e^{3x_3}}}.$$

(e) We have $\Gamma = \{(x_1, x_1) : x_1 \in \mathbb{R}\}$ and $F(p, z, x) = z p_1 + p_2 - 1$, so

$$\begin{aligned} D_p F(p,z,x) &= \begin{pmatrix} z \\ 1 \end{pmatrix}, \\ F_z(p,z,x) &= p_1, \\ D_x F(p,z,x) &= 0, \end{aligned}$$

so the characteristic equations are

$$\begin{aligned} \dot{x} &= \begin{pmatrix} z \\ 1 \end{pmatrix}, \\ \dot{z} &= p_1 z + p_2 = 1, \\ \dot{p} &= -\begin{pmatrix} p_1^2 + z \\ p_1 p_2 + 1 \end{pmatrix}. \end{aligned}$$

Then $z(s) = z^0 + s$, so $\dot{x}_1 = z^0 + s$ and $\dot{x}_2 = 1$, whence

$$x_1(s) = z^0 s + \tfrac{1}{2} s^2 + x_1^0 \qquad \text{and} \qquad x_2(s) = s + x_2^0 = s + x_1^0$$

(since $x_1^0 = x_2^0$ for $x^0 \in \Gamma$). Insert the boundary condition $z^0 = \frac{1}{2}x_1^0$ to get

$$x_1(s) = \tfrac{1}{2}s^2 + \tfrac{1}{2}x_1^0 s + x_1^0.$$

If $x \in \mathbb{R}^2$ is arbitrary then we would like to find $x^0 \in \mathbb{R}^2$ and $s \in \mathbb{R}$ such that $x_2 = s + x_1^0$, i.e. $x_1^0 = x_2 - s$, and

$$x_1 = \tfrac{1}{2}s^2 + \tfrac{1}{2}x_1^0 s + x_1^0 = \tfrac{1}{2}x_2 s + x_2 - s$$

Hence,

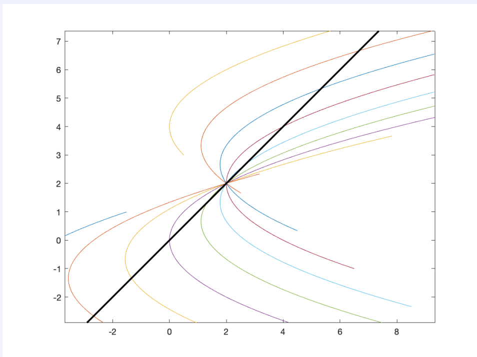
$$s = \frac{x_1 - x_2}{\frac{1}{2}x_2 - 1}.$$

Thus, we can insert into $u(x) = z(s) = z^0 + s = \frac{1}{2}x_1^0 + s = \frac{1}{2}x_2 + \frac{1}{2}s$ and get

$$u(x) = \tfrac{1}{2}x_2 + \tfrac{1}{2}\frac{x_1 - x_2}{\frac{1}{2}x_2 - 1} = \underline{\underline{\tfrac{1}{2}x_2 + \frac{x_1 - x_2}{x_2 - 2}}}.$$

It is now straightforward to check that this function solves the problem, *as long as* $x_2 \neq 2$. Thus, if we set, say, $\Omega = \{x \in \mathbb{R}^2 : 2 < x_1 < x_2\}$, then the above function solves the problem in U.

The figure below shows a selection of characteristics drawn in the vicinity of the diagonal Γ (in black):



It is clear that something strange happens at the boundary point $(2, 2)$: All characteristics join into one point! Thus, there is no unique way of moving along a characteristic starting at this point. Let us investigate what goes wrong at this point. The admissibility conditions state that $z^0 = g(x^0) = \frac{1}{2}x_1^0$ and

$$0 = F(p^0, z^0, x^0) = z^0 p_1^0 + p_2^0 - 1 = \tfrac{1}{2}x_1^0 p_1^0 p_2^0 - 1.$$

The remaining admissibility criterium is that $p^0 \cdot \tau = \tau \cdot Dg(x^0)$, where τ is any tangent vector of Γ at x^0. If, say, $\tau = (1, 1)^\top$ then $\tau \cdot Dg(x^0) \equiv \frac{1}{2}$, and $p^0 \cdot \tau = p_1^0 + p_2^0$. Thus, $p_2^0 = \frac{1}{2} - p_1^0$, which inserted into the equation for F yields

$$0 = \tfrac{1}{2} x_1^0 p_1^0 + \tfrac{1}{2} - p_1^0 - 1 \quad \Leftrightarrow \quad p_1^0 = \frac{1}{x_1^0 - 2}.$$

This clearly breaks down at $x^0 = (2, 2)$, so the admissibility condition is *not* satisfied at this point, and the theory of characteristics breaks down.