Problem set 1 – Solutions
MAT4301

Ulrik Skre Fjordholm

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Problems marked with * can be skipped if you are short on time.

Vector calculus

1. Compute the gradient Du of the following functions:

(a) $u(x) = \sin(x_1 x_2^2 - x_3)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

(b) $u(x) = |x|^2$ for $x \in \mathbb{R}^n$

(c) $u(x) = |x|$ for $x \in \mathbb{R}^n$

(Here and elsewhere, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ denotes the Euclidean norm of x.)

2. Write out the expression

$$\sum_{|\alpha|=2} \alpha! x^\alpha$$

where $x = (x_1, x_2)$ is some point in $\mathbb{R}^2$.

(Here and elsewhere we use the convention that α denotes a multiindex, so the sum runs over all pairs of nonnegative integers $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ whose sum $|\alpha| = \alpha_1 + \alpha_2$ equals 2.)

3. Compute the partial derivative $D^\alpha u$ for all multiindices α of length $|\alpha| = 1$ and $|\alpha| = 2$, for the functions $u(x) = |x|^2$ and $u(x) = |x|$ (where $x \in \mathbb{R}^n$).

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a given function, fix a point $x \in \mathbb{R}^2$ and define $g(t) = f(tx)$. Write out $g'(t)$ and $g''(t)$ in terms of partial derivatives of f. Use multiindex notation.

* **5.** Solve problems 3, 4, 5 in Section 1.5 in Evans.

Hint for 1.5.3: Use induction on n, not k. Recall the binomial theorem, $(a+b)^m = \sum_{r=0}^m \binom{m}{r} a^r b^{m-r}$, where $\binom{m}{r} = \frac{m!}{r!(m-r)!}$ are the binomial coefficients.

Hint for 1.5.4: Use induction on n. Recall Leibniz' formula in one dimension: $\partial_{x_n}^k (fg) = \sum_{r=0}^k \binom{k}{r} \partial_{x_n}^r f \, \partial_{x_n}^{k-r} g$ for functions $f, g \in C^k(\mathbb{R}^n)$.

Hint for 1.5.5: As mentioned in the exercise, define $g(t) = f(tx)$ for $t \in \mathbb{R}$. Write down the kth order Taylor expansion for $g(1)$ (including error term), expanded around $t = 0$. Show that the mth derivative of g can be written as $g^{(m)}(t) = \sum_{|\alpha|=m} \binom{m}{\alpha} x^\alpha D^\alpha f(tx)$. To this end:

- Show first that

$$g^{(m)}(t) = \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n x_{i_1} \cdots x_{i_k} \partial_{x_{i_1}} \cdots \partial_{x_{i_m}} f(tx).$$

- Next, recall the fact that for a multiindex α of length $|\alpha| = m$, the number $\binom{m}{\alpha} = \frac{m!}{\alpha_1!\cdots\alpha_n!}$ is the number of ways to extract m balls of n different colors from a bag, picking α_1 of the first color, α_2 of the second color, and so on. Use this fact to rewrite the above expression for $g^{(m)}(t)$ in multiindex notation.

Solution:

1. (a) We have $u_{x_1}(x) = x_2^2\cos(x_1x_2^2 - x_3)$, $u_{x_2}(x) = 2x_1x_2\cos(x_1x_2^2 - x_3)$ and $u_{x_3}(x) = -\cos(x_1x_2^2 - x_3)$, so

$$Du(x) = \cos(x_1x_2^2 - x_3)\begin{pmatrix} x_2^2 \\ 2x_1x_2 \\ -1 \end{pmatrix}.$$

(b) We have $u_{x_i}(x) = 2x_i$, so $Du(x) = 2x$.

(c) We have $u_{x_i}(x) = \frac{1}{2\sqrt{x_1^2+\cdots+x_n^2}}2x_i = \frac{x_i}{|x|}$, so $Du(x) = \frac{x}{|x|}$.

2. The two-dimensional multiindices of length $|\alpha| = 2$ are $(2, 0)$, $(1, 1)$ and $(0, 2)$. Therefore,

$$\sum_{|\alpha|=2} \alpha! x^\alpha = (2,0)!x^{(2,0)} + (1,1)!x^{(1,1)} + (0,2)!x^{(0,2)} = 2x_1^2 + x_1x_2 + 2x_2^2.$$

3. The partial derivatives D^α when $|\alpha| = 1$ are the components of the gradient of the function, which we already computed.

$u(x) = |x|^2$ We have $u_{x_i}(x) = 2x_i$. If $j \neq i$ then $u_{x_ix_j}(x) = 0$, while $u_{x_ix_i}(x) = 2$. Hence, we can write $D^\alpha u(x) = 2\delta_{\alpha_1,\alpha_2}$, where δ is the Krönecker delta function:

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

$u(x) = |x|$ We have $u_{x_i}(x) = \frac{x_i}{|x|}$. We have $\left(\frac{1}{|x|}\right)_{x_j} = -\frac{1}{|x|^2}|x|_{x_j} = -\frac{x_j}{|x|^3}$. Hence, if $j \neq i$ then $u_{x_ix_j}(x) = -\frac{x_ix_j}{|x|^3}$. On the other hand, for $i = j$ we get $u_{x_ix_i}(x) = \frac{1}{|x|} - \frac{x_i^2}{|x|^3}$. Summarizing, we have for any α of length $|\alpha| = 2$

$$D^\alpha u(x) = \frac{\delta_{\alpha_1,\alpha_2}}{|x|} - \frac{x_{\alpha_1}x_{\alpha_2}}{|x|^3}.$$

4. We have

$$g'(t) = f_{x_1}(tx)x_1 + f_{x_2}(tx)x_2 = x_1D^{(1,0)}f(tx) + x_2D^{(0,1)}f(tx)$$

and

$$\begin{aligned} g''(t) &= x_1^2 f_{x_1x_1}(tx) + 2x_1x_2 f_{x_1x_2}(tx) + x_2^2 f_{x_2x_2}(tx) \\ &= x_1^2 D^{(2,0)}f(tx) + 2x_1x_2D^{(1,1)}f(tx) + x_2^2D^{(0,2)}f(tx). \end{aligned}$$

Integration

1. Compute the integral

$$\int_{B(0,1)} \operatorname{div} f(x)\, dx$$

where $B(0,1)$ is the unit ball in $\mathbb{R}^3$ and $f(x) = |x|^2 x$. What do you get when $B(0,1)$ is the unit ball (or *disc*) in $\mathbb{R}^2$?
Hint: Use the divergence theorem (Theorem 1(ii) in §C.2).

2. Use the Gauss–Green theorem (Theorem 1(i) in §C.2) to prove all of the other identities in §C.2.

3. A function $u\colon \mathbb{R}^n \to \mathbb{R}$ is *locally integrable* if for every bounded set $K \subset \mathbb{R}^n$, the integral

$$\int_K |u(x)|\, dx$$

is finite.

(a) Show that $u(x) = \log|x|$ for $x \in \mathbb{R}^n$ is locally integrable, for any number of dimensions $n \in \mathbb{N}$.

(b) Let $u(x) = |x|^p$ for $x \in \mathbb{R}^n$ and let $p \in \mathbb{R}$ be a given number. For what values of p is this function locally integrable?

Hint: If u is bounded on K (i.e., $\exists\, C > 0$ such that $|u(x)| \leqslant C$ for all $x \in K$) then u is integrable over K, so it suffices to concentrate on bounded domains K where u is unbounded.

Solution:

1. We use the divergence theorem:

$$I := \int_{B(0,1)} \operatorname{div} f(x)\, dx = \int_{\partial B(0,1)} f(x)\cdot\nu(x)\, dS(x) = \int_{\partial B(0,1)} |x|^2 x\cdot x\, dS(x) = |\partial B(0,1)|$$

where we have used the fact that $\nu(x) = x$ on the unit sphere. We can look up the area $|\partial B(0,1)|$ and find the answer $I = 4\pi$.

In $\mathbb{R}^2$ we get instead $I = |\partial B(0,1)| = 2\pi$, the length of the unit circle.

2. We assume that the identity $\int_U u_{x_i}\, dx = \int_{\partial U} u\nu^i\, dS$ holds.

Divergence theorem:

$$\int_U \operatorname{div} u\, dx = \sum_{i=1}^n \int_U u^i_{x_i}\, dx = \sum_{i=1}^n \int_{\partial U} u^i \nu^i\, dS(x) = \int_{\partial U} u\cdot\nu\, dS.$$

Integration by parts:

$$\int_U u_{x_i} v\, dx = \int_U (uv)_{x_i} - uv_{x_i}\, dx = \int_{\partial U} uv\nu^i\, dS - \int_U uv_{x_i}\, dx,$$

where we in the first step have used the product rule $u_{x_i} v = (uv)_{x_i} - uv_{x_i}$.

Theorem 3(i):

$$\int_U \Delta u \, dx = \sum_{i=1}^n \int_U (u_{x_i})_{x_i} \, dx = \sum_{i=1}^n \int_{\partial U} u_{x_i} \nu^i \, dS = \int_{\partial U} Du \cdot \nu \, dS.$$

Theorem 3(ii):

$$\begin{aligned}
\int_U Du \cdot Dv \, dx &= \sum_{i=1}^n \int_U u_{x_i} v_{x_i} \, dx = \sum_{i=1}^n \int_U (u_{x_i} v)_{x_i} - u_{x_i x_i} v \, dx \\
&= \sum_{i=1}^n \int_{\partial U} u_{x_i} v \nu^i \, dS - \sum_{i=1}^n \int_U u_{x_i x_i} v \, dx \\
&= \int_{\partial U} \frac{\partial u}{\partial \nu} v dS - \int_U v \Delta u \, dx
\end{aligned}$$

where we denote $\frac{\partial u}{\partial \nu} = \nu \cdot Du$.

Theorem 3(iii):

$$\begin{aligned}
\int_U u\Delta v - v\Delta u \, dx &= \sum_{i=1}^n \int_U u v_{x_i x_i} - v u_{x_i x_i} \, dx \\
&= \sum_{i=1}^n \int_U (u v_{x_i})_{x_i} - u_{x_i} v_{x_i} - (v u_{x_i})_{x_i} + v_{x_i} u_{x_i} \, dx \\
&= \sum_{i=1}^n \int_{\partial U} u v_{x_i} \nu^i - v u_{x_i} \nu^i \, dS \\
&= \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \, dS.
\end{aligned}$$

3. (a) We only need to check that u is integrable near its singularity at $x = 0$. Let, say, $K = B(0, 1)$; then

$$\begin{aligned}
\int_K \big|\log |x|\big| \, dx &= \int_0^1 \int_{\partial B(0,r)} -\log r \, dS(x) \, dr = -\int_0^1 |\partial B(0, r)| \log r \, dr \\
&= -|\partial B(0, 1)| \int_0^1 r^{n-1} \log r \, dr,
\end{aligned}$$

which is finite for any $n \geqslant 1$. Hence, u is locally integrable. Here, we have first converted to polar coordinates and then used the fact that the (hyper-)area $|\partial B(0, r)|$ of the (hyper-)sphere $\partial B(0, r)$ equals $r^{n-1} |\partial B(0, 1)|$.

(b) If $p \geqslant 0$ then the function is locally bounded (i.e., bounded on every bounded set), so it's locally bounded. If $p < 0$ then u is bounded away from $x = 0$, so we only need to check integrability near 0. If, say $K = B(0,1)$ then

$$\begin{aligned}\int_K u(x)\,dx &= \int_0^1 \int_{\partial B(0,r)} r^p \,dS(x)dr = \int_0^1 r^p |\partial B(0,r)|\,dr \\ &= |\partial B(0,1)| \int_0^1 r^p r^{n-1}\,dr = |\partial B(0,1)| \int_0^1 r^{p+n-1}\,dr \\ &= |\partial B(0,1)| \frac{1}{p+n},\end{aligned}$$

where the last step is only true if $p+n-1 > -1$, that is, $p > -n$; otherwise, the integral is infinite.
To conclude, $u(x) = |x|^p$ is locally integrable if and only if $p > -n$.

PDEs

1. Find a function $u\colon \mathbb{R}^3 \to \mathbb{R}$ satisfying the PDE

$$-\Delta u = 1 \qquad \text{in } \mathbb{R}^3.$$

Hint: Try the function $v(x) = |x|^2$ first.

2. Solve the previous problem with $\mathbb{R}^3$ replaced by $\mathbb{R}^n$, for any $n \in \mathbb{N}$.

3. Solve problem 1.5.1 in Evans.

Solution:

1. If $v(x) = |x|^2$ then

$$\Delta v(x) = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\left(x_1^2 + x_2^2 + x_3^2\right) = 2 + 2 + 2 = 6.$$

Hence, the function $u(x) = -\frac{1}{6}|x|^2$ solves the PDE.

2. We see that $\Delta |x|^2 = 2n$, so the solution for general n is $u(x) = -\frac{1}{2n}|x|^2$.

3. Section 1.2.1:

a.1 Laplace Linear, second-order

a.2 Helmholtz Linear, second-order

a.3 Linear transport Linear, first-order

a.4 Liouville Linear, first-order

a.5 Heat Linear, second-order

a.6 Schrödinger Linear, second-order

a.7 Kolmogorov Linear, second-order

a.8 Fokker-Planck Linear, second-order

a.9 Wave Linear, second-order

a.10 Klein–Gordon Linear, second-order

a.11 Telegraph Linear, second-order

a.12 General wave Linear, second-order

a.13 Airy Linear, third-order

a.14 Beam Linear, fourth-order

b.1 Eikonal Nonlinear, first-order

b.2 Nonlinear Poisson Semilinear, second-order

b.3 p-Laplacian Quasilinear, second-order

b.4 Minimal surface Quasilinear, second-order

b.5 Monge–Ampère Nonlinear, second-order

b.6 Hamilton–Jacobi Nonlinear, first-order (provided H is nonlinear in its first argument – otherwise the equation linear)

b.7 Scalar conservation law Quasilinear, first-order (provided f is nonlinear – otherwise the equation is linear)

b.8 Inviscid Burgers equation Quasilinear, first-order

b.9 Scalar reaction-diffusion Semilinear, second-order

b.10 Porous medium Quasilinear, second-order (unless $\gamma = 1$, in which case it's linear)

b.11 Nonlinear wave equation Semilinear, second-order

b.12 KdV Semilinear, third-order

b.13 Nonlinear Schrödinger Semilinear, second-order

Section 1.2.2:

a.1 Linear elasticity, equilibrium Linear, second-order

a.2 Linear elasticity Linear, second-order

a.3 Maxwell Linear, first-order

b.1 System of conservation laws Nonlinear, first-order (unless F is linear)

b.2 Reaction-diffusion system Semilinear, second-order

b.3 Euler Quasilinear, first-order

b.4 Navier–Stokes Quasilinear, second-order