Problem set 2
MAT4301

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September 6, 2023

Problems marked with * can be skipped if you are short on time.

1. Consider the Poisson equation

$$-\Delta u = f \qquad (x \in \mathbb{R}^n). \tag{1}$$

   (a) Write out this equation in $n = 1$ dimensions.

   (b) Find all solutions of (1) in $n = 1$ dimensions when

$$f \equiv 0 \qquad \text{and} \qquad f(x) = \sin(ax) \text{ for some } a \in \mathbb{R}.$$

   (c) Solve the following *boundary value problem* on the interval $(a, b) \subset \mathbb{R}$,

$$\begin{cases} -u''(x) = 1 & (x \in (a,b)) \\ u(a) = 0,\ u(b) = 1. \end{cases}$$

2. Solve the *initial value problem*

$$\begin{cases} u_t + b \cdot Du = 0 & (x \in \mathbb{R}^2,\ t > 0) \\ u(x,0) = g(x) & (x \in \mathbb{R}^2) \end{cases} \tag{2}$$

   where $u \colon \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ is the unknown, $b = (1, 3)$ and $g(x) = \sin(x_1)\cos(x_2)$.

3. Solve the *initial-boundary value problem*

$$\begin{cases} u_t + bu_x = 0 & (x > 0,\ t > 0) \\ u(x,0) = g(x) & (x > 0) \\ u(0,t) = h(t) & (t \geqslant 0) \end{cases} \tag{3}$$

   where $u : \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}$ is the unknown, $b = 2$, $g(x) = x$ and $h(t) = t^2$.

   Hint: Draw a picture of the domain (the first quadrant of the x-t-plane) and graph the characteristics. Where do they hit the boundary of the domain?

4. Consider (3) again, but this time with a velocity $b < 0$. Does there always exist a solution of (3)?

   Hint: Draw a new picture of the domain and characteristics.