Problem set 2 – Solutions
MAT4301

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Problems marked with * can be skipped if you are short on time.

1. Consider the Poisson equation

$$-\Delta u = f \qquad (x \in \mathbb{R}^n). \tag{1}$$

(a) Write out this equation in $n = 1$ dimensions.

(b) Find all solutions of (1) in $n = 1$ dimensions when

$$f \equiv 0 \qquad \text{and} \qquad f(x) = \sin(ax) \text{ for some } a \in \mathbb{R}.$$

(c) Solve the following *boundary value problem* on the interval $(a, b) \subset \mathbb{R}$,

$$\begin{cases} -u''(x) = 1 & (x \in (a,b)) \\ u(a) = 0, \ u(b) = 1. \end{cases}$$

Solution:

(a) The equation in $n = 1$ dimensions states

$$-u''(x) = f(x) \qquad (x \in \mathbb{R}).$$

(b) If $f \equiv 0$ then $u'' \equiv 0$, that is, $u(x) = Ax + B$ for constants $A, B \in \mathbb{R}$.

If $f(x) = \sin(ax)$ then integrating twice yields

$$u(x) = Ax + B - \int^x \int^y \sin(az)\,dz\,dy = Ax + B + \frac{\sin(ax)}{a^2}.$$

(c) Integrating the equation $u''(x) = -1$ twice yields $u(x) = Ax + B - \frac{x^2}{2}$ for constants $A, B \in \mathbb{R}$. Inserting the boundary data yields

$$0 = u(a) = Aa + B - \frac{a^2}{2}, \qquad 1 = Ab + B - \frac{b^2}{2}.$$

Solving for A and B yields

$$A = \frac{2 + b^2 - a^2}{2(b-a)}, \qquad B = \frac{a^2}{2} - aA.$$

2. Solve the *initial value problem*

$$\begin{cases} u_t + b \cdot Du = 0 & (x \in \mathbb{R}^2,\ t > 0) \\ u(x,0) = g(x) & (x \in \mathbb{R}^2) \end{cases} \tag{2}$$

where $u\colon \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$ is the unknown, $b = (1,3)$ and $g(x) = \sin(x_1)\cos(x_2)$.

Solution: The method of characteristics shows that $s \mapsto u(x+sb, t+s)$ is constant, for any choice of x and t. Setting $s = 0$ and $s = -t$ then yields

$$u(x,t) = u(x - tb, 0) = g(x - tb) = \sin(x_1 - t)\cos(x_2 - 3t).$$

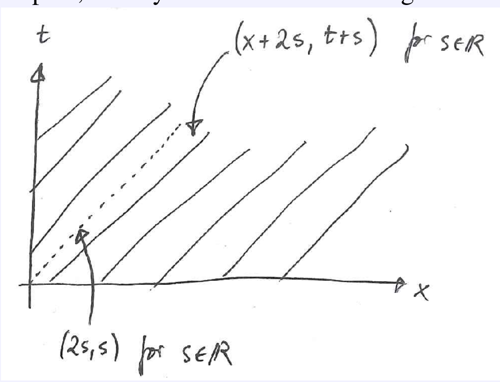
3. Solve the *initial-boundary value problem*

$$\begin{cases} u_t + bu_x = 0 & (x > 0,\ t > 0) \\ u(x,0) = g(x) & (x > 0) \\ u(0,t) = h(t) & (t \geqslant 0) \end{cases} \tag{3}$$

where $u : \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+ \to \mathbb{R}$ is the unknown, $b = 2$, $g(x) = x$ and $h(t) = t^2$.

Hint: Draw a picture of the domain (the first quadrant of the x-t-plane) and graph the characteristics. Where do they hit the boundary of the domain?

Solution: The characteristics are straight lines in the (first quadrant of the) x-t-plane with slope 2, so they look like the following:



The characteristics are of the form $s \mapsto (x + 2s, t + s)$, and u is constant along these curves. Characteristics that cross the positive x-axis take data from the initial data g, while those that cross the positive t-axis take data from the boundary data h. The boundary separating these two domains is the curve $x = t/2$. Thus, points (x,t) with $x < t/2$ take data from h, while those with $x > t/2$ take data from g. This gives

$x < t/2$**:** Set $s = 0$ and $s = -x/2$ to get

$$u(x,t) = u(0, t - x/2) = h(t - x/2) = (t - x/2)^2.$$

$x > t/2$: Set $s = 0$ and $s = -t$ to get

$$u(x,t) = u(x-2t,0) = g(x-2t) = x - 2t.$$

Along the curve $x = t/2$, these two expressions coincide, we can use either expression. We conclude that the solution is

$$u(x,t) = \begin{cases} (t - x/2)^2 & (x \leqslant t/2) \\ x - 2t & (x > t/2). \end{cases}$$

4. Consider (3) again, but this time with a velocity $b < 0$. Does there always exist a solution of (3)?

Hint: Draw a new picture of the domain and characteristics.

Solution: The characteristics now go from right to left, instead of from left to right. Thus, each characteristic will cross both the positive x and the positive t axis. Indeed, the characteristics are curves $s \mapsto (x - sb, t - s)$; setting $s = t$ and $s = x/b$ gives

$$u(x - bt, 0) = u(0, t - x/b)$$

(note that both $x - bt$ and $t - x/b$ are positive, since $b < 0$). Inserting the initial and boundary data then yields

$$g(x - bt) = h(t - x/b). \tag{$*$}$$

But g and h are two arbitrary functions, and do not necessarily satisfy the above condition! (For instance in the case $g(x) = x$, $h(t) = t^2$.) A solution of (3) for $b < 0$ therefore exists if and only if $(*)$ holds.

Remark. If $(*)$ holds, then the boundary condition automatically holds if the initial condition holds – thus, it is superfluous to require a boundary condition in this case. When $b < 0$, we can think of characteristics as flowing *out of* the domain along the boundary $\{(x,t) : x = 0\}$, while if $b > 0$ then characteristics are flowing *into* the domain. The morale of the previous two problems is that *we should not impose boundary data at outflow boundaries.*