Problem set 3
MAT4301

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1. *(Problem 2.5.4 in Evans)* Give a direct proof that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic within a bounded, open set Ω, then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Hint: Define $u_\varepsilon := u + \varepsilon|x|^2$ for $\varepsilon > 0$, and show u_ε cannot attain its maximum over $\overline{\Omega}$ at an interior point.

Hint: Prove the above by contradiction by assuming that u_ε has a maximum at some $x_0 \in \Omega$. What is then $\Delta u_\varepsilon(x_0)$?

2. *(Problem 2.5.5 in Evans)* We say $v \in C^2(\Omega)$ is *sub-harmonic* if

$$-\Delta v \leqslant 0 \qquad \text{in } \Omega. \tag{1}$$

(a) Prove for sub-harmonic v that

$$v(x) \leqslant \fint_{B(x,r)} v\,dy \qquad \text{for all } B(x,r) \subset \Omega. \tag{2}$$

(b) Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that u is harmonic and $v := \varphi(u)$. Prove v is sub-harmonic.

(c) Prove that $v := |Du|^2$ is sub-harmonic, whenever u is harmonic.

3. (a) Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be sub-harmonic within a bounded, open set Ω. Prove that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(b) Show that the same statement with "max" replaced by "min" is not necessarily true. That is, find a sub-harmonic function u on a bounded, open set Ω for which $\min_{\overline{\Omega}} u < \min_{\partial\Omega} u$.

4. (a) *(Problem 2.5.6 in Evans)* Let Ω be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on Ω, such that

$$\max_{\overline{\Omega}} |u| \leqslant C\big(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f|\big) \tag{3}$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Here, $f\colon \overline{\Omega} \to \mathbb{R}$ and $g\colon \partial\Omega \to \mathbb{R}$ are given, continuous functions.

Hint: Find an $\varepsilon > 0$ so that the function $u_\varepsilon(x) = u(x) + \varepsilon|x|^2$ from Problem **1** becomes sub-harmonic. Then do the same for the function $v_\varepsilon(x) = -u(x) + \varepsilon|x|^2$.

(b) State and prove a result which makes the following claim rigorous: If u_1 and u_2 are solutions of (4) with data f_1, g_1 and f_2, g_2, respectively, then u_1 and u_2 are close whenever the data are close.