Problem set 3 – Solutions
MAT4301

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1. *(Problem 2.5.4 in Evans)* Give a direct proof that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic within a bounded, open set Ω, then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Hint: Define $u_\varepsilon := u + \varepsilon|x|^2$ for $\varepsilon > 0$, and show u_ε cannot attain its maximum over $\overline{\Omega}$ at an interior point.

Hint: Prove the above by contradiction by assuming that u_ε has a maximum at some $x_0 \in \Omega$. What is then $\Delta u_\varepsilon(x_0)$?

Solution: Define $u_\varepsilon(x) = u(x) + \varepsilon|x|^2$. Then

$$\Delta u_\varepsilon(x) = \Delta u(x) + \varepsilon\Delta|x|^2 = 2n\varepsilon > 0.$$

Assume, by way of contradiction, that u_ε has a maximum at some point $x_0 \in \Omega$. Then $Du_\varepsilon(x_0) = 0$ and $\Delta u_\varepsilon(x_0) \leqslant 0$, which contradicts the above computation. Thus, u_ε cannot have maxima inside of Ω. Since $\overline{\Omega}$ is a closed and bounded set, and u_ε is continuous on that set, it must attain a maximum somewhere, which therefore must be on the boundary. This proves the claim.

2. *(Problem 2.5.5 in Evans)* We say $v \in C^2(\Omega)$ is *sub-harmonic* if

$$-\Delta v \leqslant 0 \qquad \text{in } \Omega. \tag{1}$$

(a) Prove for sub-harmonic v that

$$v(x) \leqslant \fint_{B(x,r)} v\,dy \qquad \text{for all } B(x,r) \subset \Omega. \tag{2}$$

(b) Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that u is harmonic and $v := \varphi(u)$. Prove v is sub-harmonic.

(c) Prove that $v := |Du|^2$ is sub-harmonic, whenever u is harmonic.

Solution:

(a) The proof proceeds just as for harmonic functions, only that instead of using $\Delta v = 0$ we use $\Delta v \geqslant 0$. Below is an outline of the proof.

Define

$$\begin{aligned}\psi(r) &:= ⨍_{B(x,r)} v(y)\,dy(y) = \frac{1}{\alpha(n)r^n}\int_{B(x,r)} v(y)\,dy \\ &= \frac{1}{\alpha(n)}\int_{B(0,1)} v(x+rz)\,dz,\end{aligned}$$

where $\alpha(n)$ is the volume of $B(0,1)$, and we have made the change of variables $y = x + rz$. Then

$$\psi'(r) = \frac{1}{\alpha(n)}\int_{B(0,1)} Dv(x+rz)\cdot z\,dz$$

(changing to polar coordinates)

$$= \frac{1}{\alpha(n)}\int_0^1 \int_{\partial B(0,s)} Dv(x+rz)\cdot z\,dS(z)\,ds$$

(observing that $\nu(z) = z/s$)

$$= \frac{1}{\alpha(n)}\int_0^1 s\int_{\partial B(0,s)} Dv(x+rz)\cdot \nu(z)\,dS(z)\,ds$$

(using the divergence theorem)

$$\begin{aligned}&= \frac{1}{\alpha(n)}\int_0^1 s\int_{B(0,s)} \operatorname{div}_z(Dv(x+rz))\,dz\,ds \\ &= \frac{1}{\alpha(n)}\int_0^1 s\int_{B(0,s)} r\Delta v(x+rz)\,dz\,ds\end{aligned}$$

(since $\Delta v \geqslant 0$)

$$\geqslant 0.$$

Hence, ψ is an increasing function, with $\lim_{r\to 0}\psi(r) = \lim_{r\to 0} ⨍_{B(x,r)} v(y)\,dy(y) = v(x)$. The conclusion (2) now follows.

(b) We have

$$v_{x_i} = \varphi'(u)u_{x_i}, \qquad v_{x_i x_i} = \varphi''(u)u_{x_i}^2 + \varphi'(u)u_{x_i x_i}.$$

Therefore,

$$\Delta v = \sum_{i=1}^n \underbrace{\varphi''(u)u_{x_i}^2}_{\geqslant 0} + \varphi'(u)u_{x_i x_i} \geqslant \sum_{i=1}^n \varphi'(u)u_{x_i x_i} = \varphi'(u)\Delta u = 0,$$

which proves our claim.

(c) We can write $v = \sum_{i=1}^n (u_{x_i})^2$. Since u is harmonic, also u_{x_i} is harmonic for each i, so by Problem (b), $(u_{x_i})^2$ is sub-harmonic. Since the sum of sub-harmonic functions is sub-harmonic, we conclude that v must be sub-harmonic.

3. (a) Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be sub-harmonic within a bounded, open set Ω. Prove that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(b) Show that the same statement with "max" replaced by "min" is not necessarily true. That is, find a sub-harmonic function u on a bounded, open set Ω for which $\min_{\overline{\Omega}} u < \min_{\partial\Omega} u$.

Solution:

(a) As in Problem **1**, let $v_\varepsilon(x) = v(x) + \varepsilon|x|^2$. Then v_ε is bounded and continuous on the compact set $\overline{\Omega}$, and $\Delta v_\varepsilon(x) = \Delta v(x) + 2n\varepsilon > 0$. Moreover, by the extremal value theorem, v_ε attains a maximum at some $x_0 \in \overline{\Omega}$. If $x_0 \in \Omega$ then $\Delta v_\varepsilon(x_0) \leqslant 0$, contradicting $\Delta v_\varepsilon(x_0) > 0$. Hence, $x_0 \in \partial\Omega$, and the conclusion follows.

(b) Let $\Omega = B^0(0,1)$, the open unit ball, and let $u(x) = |x|^2$. Then $\Delta u(x) = 2n > 0$, so u is sub-harmonic, but $\min_{\overline{\Omega}} u = 0 < \min_{\partial\Omega} u = 1$.

4. (a) *(Problem 2.5.6 in Evans)* Let Ω be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on Ω, such that

$$\max_{\overline{\Omega}} |u| \leqslant C\big(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f|\big) \tag{3}$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Here, $f\colon \overline{\Omega} \to \mathbb{R}$ and $g\colon \partial\Omega \to \mathbb{R}$ are given, continuous functions.
Hint: Find an $\varepsilon > 0$ so that the function $u_\varepsilon(x) = u(x) + \varepsilon|x|^2$ from Problem **1** becomes sub-harmonic. Then do the same for the function $v_\varepsilon(x) = -u(x) + \varepsilon|x|^2$.

(b) State and prove a result which makes the following claim rigorous: If u_1 and u_2 are solutions of (4) with data f_1, g_1 and f_2, g_2, respectively, then u_1 and u_2 are close whenever the data are close.

Solution:

(a) Let u solve (4), and define $u_\varepsilon(x) = u(x) + \varepsilon|x|^2$. Then

$$\Delta u_\varepsilon(x) = \Delta u(x) + \varepsilon\Delta|x|^2 = -f(x) + 2n\varepsilon.$$

If we choose $\varepsilon = \max_{x\in\overline{\Omega}} |f(x)|$, then $\Delta u_\varepsilon(x) \geqslant 0$, that is, u_ε is sub-harmonic. Hence, by Problem **3**,

$$\begin{aligned} u_\varepsilon(x) &\leqslant \max_{\partial\Omega} u_\varepsilon = \max_{\partial\Omega}(u(x) + \varepsilon|x|^2) \leqslant \max_{\partial\Omega} u(x) + \varepsilon \underbrace{\max_{\partial\Omega} |x|^2}_{=R} \\ &= \max_{\partial\Omega} g + \varepsilon R \end{aligned}$$

for all $x \in \Omega$, and therefore

$$u(x) = u_\varepsilon(x) - \varepsilon|x|^2 \leqslant u_\varepsilon(x) \leqslant \max_{\partial\Omega} g + \varepsilon R.$$

Next, setting $v_\varepsilon(x) = -u(x) + \varepsilon|x|^2$, we find that v_ε is sub-harmonic for the same ε, and therefore,

$$-u(x) \leqslant \max_{\partial\Omega}(-g) + \varepsilon R.$$

We conclude that

$$|u(x)| = \max(u(x), -u(x)) \leqslant \max_{\partial\Omega} |g| + R \max_{\overline{\Omega}} |f| \leqslant C\big(\max_{\partial\Omega} |g| + \max_{\in\overline{\Omega}} |f|\big)$$

where $C = \max(1, R)$.

(b) The statement is: If u_1 and u_2 are solutions of (4) with data f_1, g_1 and f_2, g_2, respectively, then

$$\max_{\overline{\Omega}} |u_1 - u_2| \leqslant C\big(\max_{\partial\Omega} |g_1 - g_2| + \max_{\overline{\Omega}} |f_1 - f_2|\big).$$

For the proof, apply Problem (a) to the function $u = u_1 - u_2$, which satisfies (4) with $f = f_1 - f_2$ and $g = g_1 - g_2$. The claim is them precisely (3).