Problem set 4
MAT4301

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September 25, 2023

1. *(Maximum principle)* Let $\Omega \subset \mathbb{R}^n$ be open and bounded and consider the *advection-diffusion problem*

$$b \cdot Du = \mu \Delta u \qquad (\text{in } \Omega) \tag{1}$$

where $b \in \mathbb{R}^n$ is a fixed vector (the *velocity*) and $\mu > 0$ is a given number (the *diffusion coefficient* or *viscosity*). Prove the maximum principle

$$u(x) \leqslant \max_{y \in \partial\Omega} u(y) \qquad \forall\, x \in \Omega \tag{2}$$

for any function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying (1).

Hint: Show first that if a function v satisfies $b \cdot Dv < \mu \Delta v$ in Ω, then v cannot have a maximum in Ω. Then let $v_\varepsilon(x) = u(x) + \varepsilon\big(|x|^2 - Mb \cdot x)$ for some $M, \varepsilon > 0$.

Remark. Consider the time-dependent equation

$$v_t + b \cdot Dv = \mu \Delta v \tag{1'}$$

for some $v = v(x,t)$. If v is a stationary solution (i.e., time-independent) then $v_t \equiv 0$ and so $u(x) := v(x,0)$ will be a solution of (1). The PDE (1') features both transport ($b \cdot Dv$) and diffusion ($\mu \Delta v$). For instance, it can model the distribution of heat $v(x,t)$ in a fluid which has heat conductivity μ and which moves through space with velocity b.

2. (a) Find a solution formula for the initial value problem

$$\begin{cases} u_t = k\Delta u & (x \in \mathbb{R}^n, t > 0) \\ u(x,0) = g(x) & (x \in \mathbb{R}^n) \end{cases} \tag{3}$$

for some $k > 0$ and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Recall: k is the heat conductivity, indicating how quickly heat is diffused through the material.

(b) What would happen for negative k?

(c) Find a solution formula for the initial value problem

$$\begin{cases} u_t(x,t) = k(t)\Delta u(x,t) & (x \in \mathbb{R}^n, t > 0) \\ u(x,0) = g(x) & (x \in \mathbb{R}^n) \end{cases} \tag{4}$$

for a continuous function $k\colon [0,\infty) \to (0,\infty)$.

3. *(Problem 2.5.12 from Evans)* Suppose u is smooth and solves $u_t = \Delta u$ in $\mathbb{R}^n \times (0, \infty)$.

(a) Show that $u^\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) Use (a) to show that $v(x,t) := x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.
Hint: Compute $\frac{\partial}{\partial \lambda} u^\lambda(x,t)$.

4. *(Problem 2.5.14 from Evans)* Write down an explicit formula for a solution of

$$\begin{cases} u_t + cu = \Delta u + f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{5}$$

where $c \in \mathbb{R}$ is a constant.

Hint: Pretend first that the Δu term is not there and multiply the equation by an integrating factor. Reduce the problem to one for which you already know the solution.

Remark. The term cu is a *reaction term* – it will "produce" or "remove" heat at a rate proportional to $u(x,t)$. For instance, u could be the temperature distribution in a reactive chemical: The reaction produces heat, and the rate of reaction is proportional to the temperature. The term f is a *source term* – it similarly "produces" or "removes" heat, but irrespective of the value of u.