Problem set 4 – Solutions
MAT4301

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1. *(Maximum principle)* Let $\Omega \subset \mathbb{R}^n$ be open and bounded and consider the *advection-diffusion problem*

$$b \cdot Du = \mu \Delta u \qquad (\text{in } \Omega) \tag{1}$$

where $b \in \mathbb{R}^n$ is a fixed vector (the *velocity*) and $\mu > 0$ is a given number (the *diffusion coefficient* or *viscosity*). Prove the maximum principle

$$u(x) \leqslant \max_{y \in \partial\Omega} u(y) \qquad \forall\, x \in \Omega \tag{2}$$

for any function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying (1).

Hint: Show first that if a function v satisfies $b \cdot Dv < \mu \Delta v$ in Ω, then v cannot have a maximum in Ω. Then let $v_\varepsilon(x) = u(x) + \varepsilon\big(|x|^2 - Mb \cdot x\big)$ for some $M, \varepsilon > 0$.

Remark. Consider the time-dependent equation

$$v_t + b \cdot Dv = \mu \Delta v \tag{1'}$$

for some $v = v(x,t)$. If v is a stationary solution (i.e., time-independent) then $v_t \equiv 0$ and so $u(x) := v(x,0)$ will be a solution of (1). The PDE (1') features both transport ($b \cdot Dv$) and diffusion ($\mu \Delta v$). For instance, it can model the distribution of heat $v(x,t)$ in a fluid which has heat conductivity μ and which moves through space with velocity b.

Solution: Assume first that $v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $b \cdot Dv < \mu \Delta v$ in Ω. If $x_0 \in \Omega$ is a maximum of v, then $Dv(x_0) = 0$ and $\Delta v(x_0) \leqslant 0$. Thus,

$$0 > b \cdot Dv(x_0) - \mu \Delta v(x_0) \geqslant 0,$$

a contradiction.
Next, as indicated in the hint, let $v_\varepsilon(x) = u(x) + \varepsilon\big(|x|^2 - Mb \cdot x\big)$ for some $\varepsilon, M > 0$. We know that $b \cdot Du - \mu \Delta u = 0$, so

$$\begin{aligned} b \cdot Dv_\varepsilon - \mu \Delta v_\varepsilon &= b \cdot Du - \mu \Delta + \varepsilon b \cdot (2x - Mb) - 2\varepsilon\mu \\ &= \varepsilon\big(2b \cdot x - M|b|^2 - 2\mu\big). \end{aligned}$$

If we let $\varepsilon > 0$ be arbitrary, and pick M so large that $2b \cdot x \leqslant M|b|^2$ for every $x \in \Omega$ (this is possible since Ω is bounded), then the above is strictly negative. Hence, v_ε cannot have a maximum in Ω; in particular,

$$v_\varepsilon(x) \leqslant \max_{\partial\Omega} v_\varepsilon.$$

Letting $\varepsilon \to 0$ and noting that $v_\varepsilon \to u$ uniformly, we conclude the proof of (2).

2. (a) Find a solution formula for the initial value problem

$$\begin{cases} u_t = k\Delta u & (x \in \mathbb{R}^n, t > 0) \\ u(x,0) = g(x) & (x \in \mathbb{R}^n) \end{cases} \tag{3}$$

for some $k > 0$ and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Recall: k is the heat conductivity, indicating how quickly heat is diffused through the material.

(b) What would happen for negative k?

(c) Find a solution formula for the initial value problem

$$\begin{cases} u_t(x,t) = k(t)\Delta u(x,t) & (x \in \mathbb{R}^n, t > 0) \\ u(x,0) = g(x) & (x \in \mathbb{R}^n) \end{cases} \tag{4}$$

for a continuous function $k\colon [0,\infty) \to (0,\infty)$.

Solution:

(a) Assume that u solves the equation and let $v(x,t) = u(x,t/k)$. Then

$$v_t(x,t) = \frac{1}{k}u_t(x,t/k) = \frac{1}{k}k\Delta u(x,t/k) = \Delta v(x,t),$$

so v solves the heat equation with initial data $v(x,0) = u(x,0) = g(x)$. We know that the function

$$v(x,t) = \big(\Phi(\cdot,t) * g\big)(x) = \frac{1}{\sqrt{4\pi t}^{\,n}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y)\,dy$$

solves the heat equation. Since $u(x,t) = v(x,kt)$, we conclude that the function

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}^{\,n}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4kt} g(y)\,dy$$

solves (3).

(b) If $k < 0$, our formula would involve the irrational term $\sqrt{4\pi k t}$, and we would be integrating e to the power of $|x-y|^2/4|k|t$, which is infinite. Thus, the formula clearly breaks down.

To interpret this, note that the change of variables $(x,t) \mapsto (x,t/k)$ — which transforms the equation to the standard heat equation — involves flipping the direction of time (t decreases instead of increases). This is an indication that solving the heat equation *backwards in time* is a bad idea.

(c) Let $\tau\colon [0,\infty) \to [0,\infty)$ be some function, to be determined, and let $v(x,t) = u(x,\tau(t))$. Then

$$v_t(x,t) = \tau'(t)u_t(x,\tau(t)) = \tau'(t)k(t)\Delta u(x,\tau(t)) = \tau'(t)k(t)\Delta v(x,t).$$

Thus, if $\tau'(t) = 1/k(t)$, that is, $\tau(t) = C + \int_0^t 1/k(s)\,ds$, then v solves the standard heat equation. Its initial data is $v(x,0) = u(x,\tau(0)) = u(x,C)$, so if we choose $C = 0$ then $\tau(0) = 0$, and $v(x,0) = u(x,0) = g(x)$. Inverting the definition of v, we get $u(x,t) = v(x,\tau^{-1}(t))$, where τ^{-1} is the inverse of τ (which we cannot compute explicitly without knowing what k is). To conclude, we get the solution formula

$$u(x,t) = \frac{1}{\sqrt{4\pi\tau^{-1}(t)}^n} \int_{\mathbb{R}^n} e^{-|x-y|^2/4\tau^{-1}(t)} g(y)\,dy.$$

3. *(Problem 2.5.12 from Evans)* Suppose u is smooth and solves $u_t = \Delta u$ in $\mathbb{R}^n \times (0,\infty)$.

(a) Show that $u^\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) Use (a) to show that $v(x,t) := x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.
Hint: Compute $\frac{\partial}{\partial\lambda} u^\lambda(x,t)$.

Solution:

(a) We have

$$u_t^\lambda(x,t) - \Delta u^\lambda(x,t) = \lambda^2 u_t(\lambda x, \lambda^2 t) - \lambda^2 \Delta u(\lambda x, \lambda^2 t) = 0.$$

(b) We follow the hint and get

$$\frac{\partial}{\partial\lambda} u^\lambda(x,t) = x \cdot Du(\lambda x, \lambda^2 t) + 2\lambda u_t(\lambda x, \lambda^2 t).$$

Comparing with the definition of v, we see that $v = \frac{\partial}{\partial\lambda} u^\lambda(x,t)\big|_{\lambda=1}$. Since u^λ solves the heat equation for any $\lambda > 0$, also its derivative with respect to λ will solve the heat equation (since we can interchange the order of differentiation). Thus, also v solves the heat equation.

4. *(Problem 2.5.14 from Evans)* Write down an explicit formula for a solution of

$$\begin{cases} u_t + cu = \Delta u + f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{5}$$

where $c \in \mathbb{R}$ is a constant.

Hint: Pretend first that the Δu term is not there and multiply the equation by an integrating factor. Reduce the problem to one for which you already know the solution.

Remark. The term cu is a *reaction term* – it will "produce" or "remove" heat at a rate proportional to $u(x,t)$. For instance, u could be the temperature distribution in a reactive chemical: The reaction produces heat, and the rate of reaction is proportional to the temperature. The term f is a *source term* – it similarly "produces" or "removes" heat, but irrespective of the value of u.

Solution: We follow the hint and multiply the PDE by the integrating factor e^{-ct} to get

$$\frac{\partial}{\partial t}\big(u(x,t)e^{-ct}\big) = e^{-ct}\Delta u(x,t) = e^{-ct} f(x,t).$$

Define $v(x,t) = u(x,t)e^{-ct}$. Then $e^{-ct}\Delta u(x,t) = \Delta v(x,t)$. Hence, if $h(x,t) := e^{-ct} f(x,t)$, we see that v solves the inhomogeneous heat equation

$$v_t = \Delta v + h$$

with initial data $v(x,0) = e^0 u(x,0) = g(x)$. Duhamel's principle yields the solution

$$v(x,t) = \frac{1}{\sqrt{4\pi t}^n}\int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/4(t-s)}}{\sqrt{4\pi(t-s)}^n} h(y,s)\,dy\,ds.$$

Finally, inserting $u(x,t) = e^{ct} v(x,t)$ and $h(x,s) = e^{-cs} f(x,s)$, we get the formula

$$u(x,t) = \frac{e^{ct}}{\sqrt{4\pi t}^n}\int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} e^{c(t-s)} \frac{e^{-|x-y|^2/4(t-s)}}{\sqrt{4\pi(t-s)}^n} f(y,s)\,dy\,ds.$$