Problem set 5
MAT4301

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1. *(Part of the proof of Theorem 6)* Let $T, \varepsilon > 0$. Show that the function

$$w(x,t) = \frac{1}{(T+\varepsilon-t)^{n/2}} e^{|x|^2/4(T+\varepsilon-t)}$$

satisfies the heat equation $w_t = \Delta w$ for $t \in (0, T]$, $x \in \mathbb{R}^n$. Note that $w(x,t)$ increases very quickly as $|x| \to \infty$.

2. Consider the Cauchy problem for the heat equation

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = 0 & \text{for } x \in \mathbb{R}. \end{cases} \tag{1}$$

Clearly, the trivial solution $u \equiv 0$ is one solution of (1), and this is also the solution we would get from the solution formula $u(t) = \Phi(\cdot, t) * u(\cdot, 0)$.

(a) Let now $\alpha > 1$ and define

$$v(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}, \qquad g(t) := \begin{cases} e^{-1/t^\alpha} & \text{for } t > 0 \\ 0 & \text{for } t \leqslant 0 \end{cases} \tag{2}$$

where $g^{(k)}$ is the k-th derivative of g. Show – either rigorously or by doing formal computations – that v also solves (1).

(b) Explain why there are infinitely many solutions to the Cauchy problem for the heat equation. How does this fit in with our "conditional uniqueness" result (Theorem 7 in Section 2.3 of Evans)?

3. Consider a *discrete random walk* in one dimension: At each point $x_i = i\Delta x$ and $t^n = n\Delta t$ (where $\Delta x, \Delta t > 0$ are given parameters and $i \in \mathbb{Z}$, $n \in \mathbb{N}_0$) we have a lump of particles which in the time interval $[t^n, t^{n+1}]$ has a probability p of moving to the left to x_{i-1}, and probability p of moving right to x_{i+1}. In particular, the probability of staying put at x_i is $1-2p$, so we need $p \in [0, 1/2]$. At time $t = 0$ and for each $i \in \mathbb{Z}$, we let $u_i^0 \geqslant 0$ be the amount of particles at position x_i.

(a) Let the distribution $(u_i^n)_{i \in \mathbb{Z}}$ at time t^n be given ($n \geqslant 0$). Explain why

$$u_i^{n+1} = pu_{i-1}^n + (1-2p)u_i^n + pu_{i+1}^n. \tag{3}$$

(b) Derive the relation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = k\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \tag{4}$$

where $k := p\frac{\Delta x^2}{\Delta t}$.

(c) Define $u_{\Delta t,\Delta x}(x_i, t^n) = u_i^n$, and extend $u_{\Delta t,\Delta x}$ to all points $(x,t) \in \mathbb{R} \times [0,\infty)$ by linear interpolation. We now wish to let $\Delta t, \Delta x \to 0$, but we have to be careful about how fast Δt goes to zero compared to Δx.

Assume that $u_{\Delta t,\Delta x}$ converges to some function u. In the following three limits, find a differential equation satisfied by u:

(i) $\Delta t, \Delta x \to 0$ such that $k := p\frac{\Delta x^2}{\Delta t} \to 0$

(ii) $\Delta t, \Delta x \to 0$ such that $k := p\frac{\Delta x^2}{\Delta t} \equiv$ const.

(iii) $\Delta t, \Delta x \to 0$ such that $k := p\frac{\Delta x^2}{\Delta t} \to \infty$.

(d) Relate the space-time scaling $\frac{\Delta x^2}{\Delta t} \equiv$ const. to what you know about the symmetries of the heat equation.

Note: Answering the above problems fully rigorously requires a lot of work, so formal explanations are enough.

4. Repeat problem **3**, but assume that in addition to particles jumping to neighbouring points, there is a production or destruction of particles. (For instance, the particles could be bacteria, and bacteria could die or reproduce.) More precisely, at each point x_i there is in the time interval $[t^n, t^{n+1}]$ a production $\Delta t\, f(x_i, t^n)$ of particles, for some function $f : \mathbb{R} \times [0,\infty) \to \mathbb{R}$.

5. Assume now that there is a local *drift* of particles at speed $b \in \mathbb{R}$. For definiteness, assume $b > 0$.

(a) Explain why the new probability of moving from x_i to x_{i+1} in the time interval $[t^n, t^{n+1}]$ is $p + \frac{\Delta t}{\Delta x}b$, the probability of moving from x_i to x_{i-1} is p, and the probability of staying put is $1 - 2p - \frac{\Delta t}{\Delta x}b$.

(b) Repeat problem **3** for particles with drift.