Problem set 5 – Solutions
MAT4301

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1. *(Part of the proof of Theorem 6)* Let $T, \varepsilon > 0$. Show that the function

$$w(x,t) = \frac{1}{(T+\varepsilon-t)^{n/2}} e^{|x|^2/4(T+\varepsilon-t)}$$

satisfies the heat equation $w_t = \Delta w$ for $t \in (0,T]$, $x \in \mathbb{R}^n$. Note that $w(x,t)$ increases very quickly as $|x| \to \infty$.

Solution: This is a matter of direct computation. The computations will be very similar to those for the fundamental solution of the heat equation.

2. Consider the Cauchy problem for the heat equation

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = 0 & \text{for } x \in \mathbb{R}. \end{cases} \tag{1}$$

Clearly, the trivial solution $u \equiv 0$ is one solution of (1), and this is also the solution we would get from the solution formula $u(t) = \Phi(\cdot,t) * u(\cdot,0)$.

(a) Let now $\alpha > 1$ and define

$$v(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}, \qquad g(t) := \begin{cases} e^{-1/t^\alpha} & \text{for } t > 0 \\ 0 & \text{for } t \leqslant 0 \end{cases} \tag{2}$$

where $g^{(k)}$ is the k-th derivative of g. Show – either rigorously or by doing formal computations – that v also solves (1).

(b) Explain why there are infinitely many solutions to the Cauchy problem for the heat equation. How does this fit in with our "conditional uniqueness" result (Theorem 7 in Section 2.3 of Evans)?

Solution:

(a) First, note that both g and all of its derivatives are bounded on $[0,\infty)$ and satisfy $|g^{(k)}(t)| \to 0$ as $t \to 0$. Hence, $v(x,t) \to 0$ as $t \to 0$. Differentiating, we have

$$v_t(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k}$$

and

$$v_{x_i}(x,t) = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k)!} 2k x^{2k-1},$$

$$v_{x_i x_i}(x,t) = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k)!} 2k(2k-1)x^{2k-2} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2(k-1))!} x^{2(k-1)}$$

$$= \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k},$$

so $v_t = v_{xx}$. The above computations are admittedly formal, and one needs to study the growth of $g^{(k)}(t)$ as $k \to \infty$ to rigorously conclude that all the above series converge absolutely.

(b) If u is a solution to the heat equation with initial data $u(x,0) = f(x)$, then the function $u(x,t) + \alpha v(x,t)$ is a solution to the same problem, for any $\alpha \in \mathbb{R}$. Thus, there are infinitely many solutions.

Since v does not satisfy the uniqueness condition $|v(x,t)| \leqslant Ae^{a|x|^2}$, this does not contradict the uniqueness result.

3. Consider a *discrete random walk* in one dimension: At each point $x_i = i\Delta x$ and $t^n = n\Delta t$ (where $\Delta x, \Delta t > 0$ are given parameters and $i \in \mathbb{Z}$, $n \in \mathbb{N}_0$) we have a lump of particles which in the time interval $[t^n, t^{n+1}]$ has a probability p of moving to the left to x_{i-1}, and probability p of moving right to x_{i+1}. In particular, the probability of staying put at x_i is $1-2p$, so we need $p \in [0, 1/2]$. At time $t = 0$ and for each $i \in \mathbb{Z}$, we let $u_i^0 \geqslant 0$ be the amount of particles at position x_i.

(a) Let the distribution $(u_i^n)_{i\in\mathbb{Z}}$ at time t^n be given ($n \geqslant 0$). Explain why

$$u_i^{n+1} = pu_{i-1}^n + (1-2p)u_i^n + pu_{i+1}^n. \tag{3}$$

(b) Derive the relation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = k\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \tag{4}$$

where $k := p\frac{\Delta x^2}{\Delta t}$.

(c) Define $u_{\Delta t,\Delta x}(x_i, t^n) = u_i^n$, and extend $u_{\Delta t,\Delta x}$ to all points $(x,t) \in \mathbb{R} \times [0,\infty)$ by linear interpolation. We now wish to let $\Delta t, \Delta x \to 0$, but we have to be careful about how fast Δt goes to zero compared to Δx.

Assume that $u_{\Delta t,\Delta x}$ converges to some function u. In the following three limits, find a differential equation satisfied by u:

(i) $\Delta t, \Delta x \to 0$ such that $k := p\frac{\Delta x^2}{\Delta t} \to 0$

(ii) $\Delta t, \Delta x \to 0$ such that $k := p\frac{\Delta x^2}{\Delta t} \equiv$ const.

(iii) $\Delta t, \Delta x \to 0$ such that $k := p\frac{\Delta x^2}{\Delta t} \to \infty$.

(d) Relate the space-time scaling $\frac{\Delta x^2}{\Delta t} \equiv$ const. to what you know about the symmetries of the heat equation.

Note: Answering the above problems fully rigorously requires a lot of work, so formal explanations are enough.

Solution:

(a) A particle at position i can jump to $i-1$, jump to $i+1$, or stay at i. The probabilities of these are p, p and $1-2p$, respectively. Thus, at the next timestep t^{n+1}, there will be pu_{i-1}^n particles coming from $i-1$ to i, pu_{i+1}^n particles coming from $i+1$, and $(1-2p)u_i^n$ particles staying at i. Once added up, we get (3).

(b) This is a simple reordering of (3).

(c) Note first that for fixed (x,t), if i,n are such that $x_i = x$ and $t^n = t$, then $\frac{u_i^{n+1}-u_i^n}{\Delta t} \to u_t(x,t)$ and $\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \to u_{xx}(x,t)$ as $\Delta t, \Delta x \to 0$.

  (i) If $k \to 0$ as $\Delta t, \Delta x \to 0$ then the limit satisfies $u_t(x,t) = 0$ for all x,t, that is, $u(x,t) = u_0(x)$ for all x,t.

  (ii) If $k \equiv$ const. then the limit satisfies $u_t = ku_{xx}$.

  (iii) If $k \to \infty$ then, dividing (4) by k shows that $u_{xx}(x,t) = 0$ for all x,t.

(d) We know that if u solves the heat equation then so does $u^\lambda(x,t) := u(\lambda x, \lambda^2 t)$. Letting $\lambda \to 0$ is akin to "zooming in" on u and observing finer details of u. Thus, the "correct" way of zooming in on a solution to the heat equation is by scaling time by λ^2 when scaling space by a factor λ.

Similarly, in (4), letting $\Delta x, \Delta t \to 0$ is akin to observing finer details of u (by computing a better approximation). If we set $\Delta x = \lambda$, then the scaling $p\frac{\Delta x^2}{\Delta t} \equiv$ const. is equivalent to stating that $\Delta t = c\lambda^2$ for some constant $c > 0$. Thus, just as for the heat equation, we scale time by λ^2 whenever we scale space by a factor λ.

4. Repeat problem **3**, but assume that in addition to particles jumping to neighbouring points, there is a production or destruction of particles. (For instance, the particles could be bacteria, and bacteria could die or reproduce.) More precisely, at each point x_i there is in the time interval $[t^n, t^{n+1}]$ a production $\Delta t\, f(x_i, t^n)$ of particles, for some function $f: \mathbb{R} \times [0,\infty) \to \mathbb{R}$.

Solution:

(a) At position i, from time n to time $n+1$, we now have an amount pu_{i-1}^n coming from $i-1$, an amount pu_{i+1}^n coming from $i+1$, an amount $(1-2p)u_i^n$ staying at i, and a production $\Delta t\, f(x_i, t^n)$ of new particles. Thus,

$$u_i^{n+1} = pu_{i-1}^n + (1-2p)u_i^n + pu_{i+1}^n + \Delta t\, f(x_i, t^n).$$

(b) Rearranging gives

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = k\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + f(x_i, t^n)$$

where $k := p\frac{\Delta x^2}{\Delta t}$.

(c) In the three limits we get

(i) $u_t = f$

(ii) $u_t = ku_{xx} + f$

(iii) $u_{xx} = 0$

5. Assume now that there is a local *drift* of particles at speed $b \in \mathbb{R}$. For definiteness, assume $b > 0$.

(a) Explain why the new probability of moving from x_i to x_{i+1} in the time interval $[t^n, t^{n+1}]$ is $p + \frac{\Delta t}{\Delta x} b$, the probability of moving from x_i to x_{i-1} is p, and the probability of staying put is $1 - 2p - \frac{\Delta t}{\Delta x} b$.

(b) Repeat problem **3** for particles with drift.

Solution:

(a) In a time interval of length Δt, the particles at x_i on average move a distance $\Delta t b$, which is a fraction of $\frac{\Delta t}{\Delta x} b$ of the distance between x_i and x_{i+1}. (The particles move to the right, not to the left.) In this way we can say that an amount of $\frac{\Delta t}{\Delta x} b u_i^n$ particles move from x_i to x_{i+1} in the time interval $[t^n, t^{n+1}]$. The probability of moving from x_i to x_{i-1} is still p (none of the particles can go left due to drift, since $b > 0$), and then there is a fraction of $1 - 2p - \frac{\Delta t}{\Delta x} b$ left.

(b) We get

$$u_i^{n+1} = (p + b\tfrac{\Delta t}{\Delta x} b) u_{i-1}^n + (1 - 2p - \tfrac{\Delta t}{\Delta x} b) u_i^n + p u_{i+1}^n$$

which can be rearranged as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + b\frac{u_i^n - u_{i-1}^n}{\Delta x} = k\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}.$$

In the three limits we get

(i) $u_t + bu_x = 0$

(ii) $u_t + bu_x = ku_{xx}$

(iii) $u_{xx} = 0$