Problem set 6
MAT4301

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1. (a) Write down d'Alembert's formula for the solution of the one-dimensional wave equation with wave speed $c > 0$,

$$\begin{cases} u_{tt} = c^2 u_{xx} & (x \in \mathbb{R},\ t > 0) \\ u(x,0) = g(x) \\ u_t(x,0) = h(x). \end{cases} \tag{1}$$

Draw the backwards wave cone from a point $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

Remark. The backwards wave cone at (x,t) is the set of points (y,s) with $s < t$ such that the solution at (x,t) depends on the solution at (y,s).

(b) Solve (1) when $c = 1$, $g(x) = \sin(x)$, $h(x) = \cos(2x)$.

2. Consider the one-dimensional wave equation on $\mathbb{R}_+$ with *Neumann boundary condition,*

$$\begin{cases} u_{tt} = c^2 u_{xx} & (x > 0,\ t > 0) \\ u(x,0) = g(x) & (x > 0) \\ u_t(x,0) = h(x) & (x > 0) \\ u_x(0,t) = 0 & (t > 0). \end{cases} \tag{2}$$

Where g, h are given C^1 functions satisfying $g'(0) = h'(0) = 0$.

(a) Find a solution formula for (2).
Hint: See [p. 69, Evans]. Make an even *reflection.*

(b) Make a physical interpretation of the boundary condition and of your solution formula.

3. In this problem we generalize Duhamel's principle to higher-order time-dependent equations. Consider the following second-order linear differential equation: Find $u\colon [0,\infty) \to X$ satisfying

$$\begin{cases} u''(t) = L_0 u(t) + L_1 u'(t) + f(t) & (t > 0) \\ u(0) = u_0 \\ u'(0) = u_1 \end{cases} \tag{3}$$

for some given inhomogeneity $f\colon (0,\infty) \to X$, initial data $u_0, u_1 \in X$, linear operators $L_0, L_1\colon X \to X$, and where X is some vector space (you can put $X = \mathbb{R}$ for simplicity). Assume that you are able to solve the corresponding homogeneous problem

$$\begin{cases} u''(t) = L_0 u(t) + L_1 u'(t) & (t > 0) \\ u(0) = u_0 \\ u'(0) = u_1 \end{cases} \tag{4}$$

and denote the solution by $\varphi(t; u_0, u_1)$ (so that the function $u(t) = \varphi(t; u_0, u_1)$ solves (4)).

(a) Show that u defined by

$$u(t) := \int_0^t \varphi(t-s; 0, f(s))\, ds \tag{5}$$

solves the inhomogeneous problem (3) with initial data $u_0 = u_1 = 0$.

(b) Which step of your computation would fail if the operators L_0, L_1 were nonlinear?

(c) Deduce a formula for the solution of the general problem (3).

(d) Generalize Duhamel's principle to the general k-th order linear equation

$$\begin{cases} u^{(k)}(t) = \sum_{j=0}^{k-1} L_j u^{(j)}(t) + f(t) & (t > 0) \\ u^{(j)}(0) = u_j & (j = 0, \dots, k-1) \end{cases} \tag{6}$$

where $u^{(j)}$ (for $j = 0, 1, \dots, k$) are derivatives of u, and L_j are linear operators from X to X.