MAT4360 - Fall 2017 - Exercises for Monday 02.10

We will need the following definition. Let $\mathcal{A}$ be C^*-algebra. A positive linear functional φ on $\mathcal{A}$ is called *faithful* whenever we have

$$\{A \in \mathcal{A} \mid \varphi(A^*A) = 0\} = \{0\}.$$

Exercise 14.

Let Ω be a compact Hausdorff space and set $\mathcal{A} = C(\Omega)$. Let μ be a finite regular Borel measure on Ω and consider the associated positive linear functional φ_μ on $\mathcal{A}$, i.e.,

$$\varphi_\mu(f) = \int_\Omega f \, \mathrm{d}\mu \quad \text{for all } f \in \mathcal{A}.$$

Set $\mathrm{supp}(\mu) = \{\omega \in \Omega \mid \mu(U) > 0 \text{ for every open neigbourhood } U \text{ of } \omega\}$.

Show that $\mathrm{supp}(\mu)$ is a closed subset of Ω, and that φ_μ is faithful if and only if $\mathrm{supp}(\mu) = \Omega$.

Exercise 15.

a) Assume H is a Hilbert space and $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(H)$. Let $\xi \in H$ and consider the positive linear functional φ_ξ on $\mathcal{A}$ given by $\varphi_\xi(A) = \langle A\xi, \xi\rangle$ for all $A \in \mathcal{A}$. Check that φ_ξ is faithful if and only if ξ is *separating* for $\mathcal{A}$, i.e., it satisfies $\{A \in \mathcal{A} \mid A\xi = 0\} = \{0\}$.

Consider now the case where $H = \mathbb{C}^3$ and

$$\mathcal{A} = \big\{[a_{ij}] \in M_3(\mathbb{C}) \mid a_{13} = a_{23} = a_{31} = a_{32} = 0\big\}.$$

(Here, as usual, we identify $\mathcal{B}(H)$ with $M_3(\mathbb{C})$. We also note that $\mathcal{A} \simeq M_2(\mathbb{C}) \oplus \mathbb{C}$.)

b) Let $\xi \in \mathbb{C}^3$. Show that φ_ξ is not faithful on $\mathcal{A}$.

c) Set $\xi_1 = (1, 0, 1)$ and $\xi_2 = (0, 1, 0)$. Check that $\varphi_{\xi_1} + \varphi_{\xi_2}$ is faithful on $\mathcal{A}$. Moreover, find $t_1, t_2 > 0$ such that $t_1 \varphi_{\xi_1} + t_2 \varphi_{\xi_2}$ is a faithful state on $\mathcal{A}$.

Exercise 16.

A linear functional τ on an algebra $\mathcal{A}$ is said to be *tracial* when it satisfies that $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{A}$.

Let $\mathcal{A}$ be a unital C^*-algebra and let τ be a linear functional on $\mathcal{A}$. Show that τ is tracial if and only if $\tau(UAU^*) = \tau(A)$ for all $A \in \mathcal{A}$ and all $U \in \mathcal{U}(\mathcal{A})$.

Exercise 17.

Set $\mathcal{A} = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ and define a linear functional Tr on $\mathcal{A}$ by $\mathrm{Tr}(A) = \sum_{j=1}^n a_{jj}$ for $A = [a_{ij}] \in \mathcal{A}$.

a) Show that Tr is a faithful tracial positive linear functional on $\mathcal{A}$.

b) Let $\{\xi_1, \ldots, \xi_n\}$ be an o.n.b. for $\mathbb{C}^n$ and $A \in \mathcal{A}$. Show that $\mathrm{Tr}(A) = \sum_{j=1}^n \langle A\xi_j, \xi_j\rangle$.

c) For each $S \in \mathcal{A}$ define a linear functional on $\varphi_S : \mathcal{A} \to \mathbb{C}$ by $\varphi_S(A) = \mathrm{Tr}(AS)$ for $A \in \mathcal{A}$. Show that the map $S \to \varphi_S$ is a vector space isomorphism from $\mathcal{A}$ onto $\mathcal{A}^*$. Then verify that $\varphi_S \geq 0$ if and only if $S \in \mathcal{A}^+$, in which case we have $\|\varphi_S\| = \mathrm{Tr}(S)$. Deduce that φ_S is a state on $\mathcal{A}$ if and only if $S \in \mathcal{A}^+$ and $\mathrm{Tr}(S) = 1$.

d) Assume φ is a tracial positive linear functional on $\mathcal{A}$. Show that $\varphi = t\,\mathrm{Tr}$ for some $t \geq 0$. Deduce that $\mathcal{A}$ has exactly one tracial state, namely $\tau = \frac{1}{n}\mathrm{Tr}$.