MAT4360 - Fall 2017 - Exercises for Monday 06.11

Exercise 27.

Let $(X, \mathcal{B}, \mu)$ denote a finite measure space. For $p = 2$ (resp. $p = \infty$), we consider elements of $\mathcal{L}^p(\mu)$ as equivalence classes of the form $[\xi]$ (resp. $[f]$), where ξ (resp. f) is a $\mathcal{B}$-measurable complex function on X such that $\int_X |\xi|^2 \, \mathrm{d}\mu < \infty$ (resp. such that f is essentially bounded); so $[\xi]$ (resp. $[f]$) consists of all $\mathcal{B}$-measurable complex functions on X agreeing with ξ (resp. f) μ-almost everywhere.
As known from previous courses, $\mathcal{L}^2(\mu)$ is a Hilbert space, while $\mathcal{L}^\infty(\mu)$ is a Banach space. One easily checks that $\mathcal{L}^\infty(\mu)$ is an abelian C^*-algebra w.r.t. to the natural operations $[f] \cdot [g] = [fg]$ and $[f]^* = \big[\, \overline{f} \,\big]$.

a) Let the map $M : \mathcal{L}^\infty(\mu) \to \mathcal{B}(\mathcal{L}^2(\mu))$ be given by $M(f)\,[\xi] = [f\xi]$ for $f \in \mathcal{L}^\infty(\mu)$ and $\xi \in \mathcal{L}^2(\mu)$. Show that M is a well-defined faithful $*$-representation of $\mathcal{L}^\infty(\mu)$.

b) Set $\mathcal{M} = M\big(\mathcal{L}^\infty(\mu)\big)$. Show that $\mathcal{M}' = \mathcal{M}$. This implies that $\mathcal{M}$ is a von Neumann algebra on $L^2(\mu)$. Show also that if $\mathcal{N}$ is any abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{L}^2(\mu))$ such that $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{N} = \mathcal{M}$.
(One therefore says that $\mathcal{M}$ is a *maximal abelian von Neuman subalgebra of* $\mathcal{B}(\mathcal{L}^2(\mu))$.

Exercise 28.

Let H be a Hilbert space and let $f : \mathbb{R} \to [-1, 1]$ denote the function given by

$$f(x) = \frac{2x}{1 + x^2} \,.$$

Show that the map $S \to f(S)$ from $\mathcal{B}(H)_{\mathrm{sa}}$ into itself is SOT-SOT continuous.
That is, show that if SOT-$\lim_\alpha S_\alpha = S$ for a net $\{S_\alpha\}$ in $\mathcal{B}(H)_{\mathrm{sa}}$ and some S in $\mathcal{B}(H)_{\mathrm{sa}}$, then SOT-$\lim_\alpha f(S_\alpha) = f(S)$.