MAT4360 - Fall 2017 - Exercises for Monday 09.10

Exercise 18.

Let $\mathcal{A}$ be C^*-algebra.

a) For $\varphi \in \mathcal{A}^*$, define $\varphi^* : \mathcal{A} \to \mathbb{C}$ by $\varphi^*(A) = \overline{\varphi(A^*)}$. Check that $\varphi^* \in \mathcal{A}^*$, $(\varphi^*)^* = \varphi$ and $\|\varphi^*\| = \|\varphi\|$.

b) Let $\varphi, \psi \in \mathcal{A}^*$ and $\lambda \in \mathbb{C}$. Check that $(\varphi + \psi)^* = \varphi^* + \psi^*$ and $(\lambda\varphi)^* = \overline{\lambda}\,\varphi^*$.

Exercise 19. (NB: This exercise will be part of the compulsory assignment this spring).

Let $\{\mathcal{A}_j\}_{j\in J}$ denote a family of C^*-algebras. Set

$$\mathcal{A} = \Big\{\{A_j\} \in \prod_{j\in J} \mathcal{A}_j \;:\; \sup_{j\in J} \|A_j\| < \infty\Big\}.$$

Define operations on $\mathcal{A}$ by

$$\{A_j\} + \{B_j\} = \{A_j + B_j\},\ \lambda\{A_j\} = \{\lambda A_j\},\ \{A_j\}\cdot\{B_j\} = \{A_jB_j\},\ \text{and } \{A_j\}^* = \{(A_j)^*\}$$

for $\{A_j\}, \{B_j\} \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. You can take as granted that these operations are well-defined and make $\mathcal{A}$ into a $*$-algebra.

a) For $\{A_j\} \in \mathcal{A}$, set

$$\|\{A_j\}\| = \sup_{j\in J} \|A_j\|\,.$$

It is straightforward to check that this gives a submultiplicative norm on $\mathcal{A}$ (and you are not asked to do this). Show that $\mathcal{A}$ becomes a C^*-algebra.

Note: It is common to denote $\mathcal{A}$ by $\bigoplus_{j\in J} \mathcal{A}_j$ and call it the *direct sum* of the family $\{\mathcal{A}_j\}_{j\in J}$. If $J = \{1, 2, \dots, n\}$, then one denotes $\mathcal{A}$ by $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$. If $\mathcal{A}_j = \mathcal{B}$ for all $j \in J$, one usually writes $\mathcal{A} = \ell^\infty(J, \mathcal{B})$.

b) Set $\mathcal{A}_0 = \Big\{\{A_j\} \in \mathcal{A} \,:\, \forall\, \varepsilon > 0\,, \exists\, F \subseteq J\,,\ F \text{ finite, such that } \|A_j\| < \varepsilon\ \forall\, j \in J \setminus F\Big\}$. Show that $\mathcal{A}_0$ is a closed (two-sided) ideal of $\mathcal{A}$.

Note: It is common to denote $\mathcal{A}_0$ by $\bigoplus^{c_0}_{j\in J} \mathcal{A}_j$ and call it the *restricted direct sum* of the family $\{\mathcal{A}_j\}_{j\in J}$. If $\mathcal{A}_j = \mathcal{B}$ for all $j \in J$, one usually writes $\mathcal{A}_0 = c_0(J, \mathcal{B})$.

Exercise 20.

Let $\{H_j\}_{j\in J}$ denote a family of Hilbert spaces. Set

$$H = \Big\{\xi = \{\xi_j\} \in \prod_{j\in J} H_j \;:\; \sum_{j\in J} \|\xi_j\|^2 < \infty\Big\}.$$

a) Let $\xi = \{\xi_j\}, \eta = \{\eta_j\} \in H$. Show that the series $\sum_{j\in J} \langle \xi_j, \eta_j\rangle$ is convergent. Denote the sum of this series by $\langle \xi, \eta\rangle$.

b) Check that H becomes a Hilbert space w.r.t. $\langle\cdot,\cdot\rangle$. It is called the (Hilbert) *direct sum* of the family $\{H_j\}_{j\in J}$ and is usually denoted by $\bigoplus_{j\in J} H_j$.

c) For each $j \in J$, let $T_j \in \mathcal{B}(H_j)$, and assume that $\sup_{j\in J} \|T_j\| < \infty$. For $\xi = \{\xi_j\} \in H$, set

$$T(\xi) = \{T_j(\xi_j)\} \in \prod_{j\in J} H_j\,.$$

Verify that $T(\xi) \in H$. Then show that the map $T : H \to H$ sending each ξ to $T(\xi)$ is a bounded operator s.t. $\|T\| = \sup_{j\in J} \|T_j\|$. We will denote T by $\widetilde{\oplus}_{j\in J}\, T_j$.

d) For each $j \in J$, let $\mathcal{A}_j$ be a C^*-subalgebra of $\mathcal{B}(H_j)$, and set $\mathcal{A} = \bigoplus_{j \in J} \mathcal{A}_j$ (cf. Exercise 19). Define a map $\Pi : \mathcal{A} \to \mathcal{B}(H)$ by

$$\Pi\big(\{T_j\}\big) = \widetilde{\oplus}_{j \in J}\, T_j$$

for each $\{T_j\} \in \mathcal{A}$. Show that Π is an isometric $*$-homomorphism.