MAT4360 - Fall 2017 - Exercises for Monday 28.08 and Monday 11.09

We let H denote a nontrivial (complex) Hilbert space and recall that $\mathcal{K}(H)$ denotes the compact operators on H, while $\mathcal{F}(H)$ denotes the bounded finite rank operators on H. By an operator we always mean a linear operator. If $\mathcal{A}$ is a nonunital C^*-algebra, then we denote by $\widetilde{\mathcal{A}}$ the C^*-algebra obtained by adjoining a unit to $\mathcal{A}$.

Exercises for Monday 28.08:

Exercise 1. Let $T : H \to H$ be a (linear) operator.

a) We have seen that if $T \in \mathcal{B}(H)$, then T is weak-weak continuous. Show that the converse statement holds.

b) Show that $T \in \mathcal{B}(H)$ if and only if T is norm-weak continuous.

NB: See also Exercise 4 !

Exercise 2. Show that $\mathcal{K}(H)$ is unital if and only if H is finite dimensional.

Exercise 3. Show that $C_0(\mathbb{R})$ is nonunital.

Exercises for Monday 11.09:

Exercise 4. Let T be as in Exercise 1. We have seen that $T \in \mathcal{K}(H)$ if and only if the map $T_{|H_1} : H_1 \to H$ is weak-norm continuous.

Show that $T \in \mathcal{F}(H)$ if and only if T is weak-norm continuous.

Exercise 5. Assume that H is infinite dimensional.

Show that $\mathcal{K}(H) + \mathbb{C}\, I_H$ is a C^*-subalgebra of $\mathcal{B}(H)$ which is isometrically $*$-isomorphic to $\widetilde{\mathcal{K}(H)}$.

Exercise 6. Let Ω denote a locally compact Hausdorff space and let $C_0(\Omega)$ denote the C^*-subalgebra of $C_b(\Omega)$ consisting of all continuous functions $f : \Omega \to \mathbb{C}$ vanishing at infinity, i.e., satisfying that for every $\varepsilon > 0$ there exists a compact subset K of Ω such that $|f(\omega)| < \varepsilon$ for all $\omega \in \Omega \setminus K$.

a) Show that $C_0(\Omega)$ is unital if and only if Ω is compact.

b) Assume that Ω is not compact. Then show that $\widetilde{C_0(\Omega)}$ is isometrically $*$-isomorphic to $C\big(\widetilde{\Omega}\big)$, where $\widetilde{\Omega}$ denotes the one-point compactification of Ω.