# MAT4360 - Fall 2017 - Exercises for Monday 18.09

In the following exercises $\mathcal{A}$ denotes a $C^*$-algebra. Whenever appropriate you may use that

$$\mathcal{A}^+ = \{B^*B : B \in \mathcal{A}\}.$$

(We will start on Monday by showing this result.)

**Exercise 7**. Assume $\mathcal{A}$ is unital, with unit $I$.

a) Let $A \in \mathcal{A}_{\mathrm{sa}}$ and let $f, g$ be real-valued continuous functions on $\mathrm{sp}(A)$. It follows from the continuous functional calculus that $f(A), g(A) \in \mathcal{A}_{\mathrm{sa}}$. Show that

$$f(A) \leq g(A) \quad \Longleftrightarrow \quad f \leq g.$$

Deduce that

$$-\|A\|\, I \ \leq \ A \ \leq \ \|A\|\, I\,.$$

b) Let $X \in \mathcal{A}$. Show that $X^*X \leq I \Longleftrightarrow \|X\| \leq 1$.

**Exercise 8**. Let $A, B \in \mathcal{A}^+$. Show that

$$A \leq B \ \Rightarrow \ \|A\| \ \leq \ \|B\|\,.$$

**Exercise 9**. Let $A, B \in \mathcal{A}$. Show that

$$\mathrm{sp}(AB) \cup \{0\} = \mathrm{sp}(BA) \cup \{0\}\,.$$

**Exercise 10**. Let $A, B \in \mathcal{A}^+$. Show that $AB \in \mathcal{A}^+ \Longleftrightarrow AB = BA$.

**Exercise 11**. Let $A, B \in \mathcal{A}_{\mathrm{sa}}$ and $C \in \mathcal{A}$. Show that

$$A \leq B \ \Rightarrow \ C^*AC \ \leq \ C^*BC\,.$$