MAT4360 - Fall 2017 - Exercises for Monday 23.10

We recall that if S is a subset of a normed space, then we let $\big[S\big]$ denote the norm-closure of the span of S. When $\mathcal{A}$ is a C^*-algebra, $\pi \in \mathrm{Rep}(\mathcal{A}, H)$, and K is closed subspace of H which is π-invariant, we denote by $\pi_{|K}$ the representation of $\mathcal{A}$ on K obtained by restriction, i.e., $\pi_{|K}$ is given by $\pi_{|K}(A)\,\eta = \pi(A)\,\eta$ for all $A \in \mathcal{A}$ and all $\eta \in K$.

Exercise 21. (NB: This exercise will be part of the compulsory assignment this spring).

Let $\mathcal{A}$ be C^*-algebra and let $\pi \in \mathrm{Rep}(\mathcal{A}, H)$ for some Hilbert space H. Set $H_\pi = \big[\pi(\mathcal{A})H\big]$. Since H_π is π-invariant, we can form $\pi^{\mathrm{ess}} \in \mathrm{Rep}(\mathcal{A}, H_\pi)$ by restriction, i.e., $\pi^{\mathrm{ess}} := \pi_{|H_\pi}$. Let $\{e_\alpha\}$ be an approximate unit for $\mathcal{A}$.

Show that $\pi^{\mathrm{ess}}(e_\alpha) \to I_{H_\pi}$ in the strong operator topology of $\mathcal{B}(H_\pi)$. Then deduce that π is nondegenerate if and only if $\pi(e_\alpha) \to I_H$ in the strong operator topology of $\mathcal{B}(H)$.

Exercise 22. (NB: This exercise will also be part of the compulsory assignment this spring).

Let $\mathcal{A}$ be a C^*-algebra.

a) Assume $\{H_j\}_{j\in J}$ is a family of Hilbert spaces and $\pi_j \in \mathrm{Rep}(\mathcal{A}, H_j)$ for all $j \in J$. Let $H' = \bigoplus_{j\in J} H_j$ denote the Hilbert direct sum of the H_j's and let $\pi' = \oplus_{j\in J}\, \pi_j \in \mathrm{Rep}(\mathcal{A}, H')$ denote the direct sum of the π_j's. For each $k \in J$, set

$$H'_k = \big\{\, \{\xi_j\} \in H' \mid \xi_j = 0 \text{ for all } j \in J \setminus \{k\} \,\big\}.$$

Check the following:

(i) Each H'_k is a closed subspace of H', and $H'_k \perp H'_l$ for all $k, l \in J$ such that $k \neq l$.

(ii) $H' = \Big[\bigcup_{k\in J} H'_k\Big]$.

(iii) For each $k \in J$, H'_k is π'-invariant, and the restriction $\pi'_k := \pi'_{|H'_k} \in \mathrm{Rep}(\mathcal{A}, H'_k)$ is unitarily equivalent to π_k.

b) Let $\pi \in \mathrm{Rep}(\mathcal{A}, H)$ for some Hilbert space H, and let $\{H_j\}_{j\in J}$ denote a family of π-invariant closed subspaces of H satisfying

$$H_k \perp H_l \text{ for all } k, l \in J \text{ such that } k \neq l, \text{ and } H = \Big[\bigcup_{k\in J} H_k\Big].$$

For each $j \in J$, set $\pi_j := \pi_{|H_j} \in \mathrm{Rep}(\mathcal{A}, H_j)$.

Show that π is unitary equivalent to $\oplus_{j\in J}\, \pi_j$. That is, show that there exists a unitary operator $U : H \to \bigoplus_{j\in J} H_j$ such that

$$U\,\pi(A)\,U^* = (\oplus_{j\in J}\, \pi_j)(A) \quad \text{for all } A \in \mathcal{A}.$$

Exercise 23.

Let $\mathcal{A}$ be C^*-algebra and let φ be a positive linear functional on $\mathcal{A}$.

a) Let $\mathcal{N}$ denote the subspace of $\mathcal{A}$ given by $\mathcal{N} = \{B \in \mathcal{A} : \varphi(B^*B) = 0\}$ and set $H_0 = \mathcal{A}/\mathcal{N}$. Write $\dot{A} = A + \mathcal{N} \in H_0$ when $A \in \mathcal{A}$. We recall that $\|\dot{A}\|_0 := \varphi(A^*A)$ gives a norm on H_0.

In the proof of the GNS-construction, we did not prove that if $\{e_\alpha\}$ is an approximate unit for $\mathcal{A}$, then $\{\dot{e_\alpha}\}$ is a Cauchy net in H_0. Do this !

(As we used in the proof, it then follows that $\{\dot{e_\alpha}\}$ converges to some $\xi_\varphi \in H_\varphi :=$ the completion of H_0.)

b) Assume that $\pi' \in \mathrm{Rep}(\mathcal{A}, H')$ is cyclic, with a cyclic vector $\xi' \in H'$ satisfying that

$$\varphi(A) = \big\langle \pi'(A)\,\xi', \xi' \big\rangle \quad \text{for all } A \in \mathcal{A}.$$

Show that there exists a unitary operator $U : H_\varphi \to H'$ such that

$$U\,\xi_\varphi = \xi' \quad \text{and} \quad U\,\pi_\varphi(A)\,U^* = \pi'(A) \;\; \text{for all } A \in \mathcal{A}.$$

(In particular, this shows that π' is unitarily equivalent to π_φ).