# MAT4360 - Fall 2017 - Exercises for Monday 25.09

**Exercise 12.**

a) For each $n \in \mathbb{N}$, pick any function $e_n \in C_0(\mathbb{R})$ taking its values in $[0,1]$, and such that $e_n$ is constantly equal to 1 on $[-n, n]$ while it is constantly equal to 0 outside $[-(n+1), n+1]$.
Show that $\{e_n\}_{n\in\mathbb{N}}$ is an approximate unit for $C_0(\mathbb{R})$.

b) Let $H$ be a Hilbert space and let $\{u_j\}_{j\in J}$ be an orthonormal basis for $H$.
Set $\mathcal{F} = \{F \subseteq J : F \text{ is nonempty and finite}\}$ and consider $\mathcal{F}$ as a directed set w.r.t. set-inclusion. For each $F \in \mathcal{F}$, let $p_F$ denote the orthogonal projection from $H$ onto the finite dimensional subspace $H_F = \text{span}\{u_j : j \in F\}$. Show that $\{p_F\}_{F\in\mathcal{F}}$ is an approximate unit for $\mathcal{K}(H)$.

**Exercise 13.**

When $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras, one may define their direct sum $\mathcal{A} \oplus \mathcal{B}$ by equipping $\mathcal{A} \times B$ with the obvious pointwise operations, and thereby obtain a $*$-algebra which contains (copies of) $\mathcal{A}$ and $\mathcal{B}$ as (two-sided) ideals. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras, then $\mathcal{A} \oplus \mathcal{B}$ becomes a $C^*$-algebra with respect to the norm $\|(A, B)\| := \max(\|A\|, \|B\|)$.

a) Let $\mathcal{C}$ be a $*$-algebra and $\mathcal{A}$, $\mathcal{B}$ be $*$-subalgebras of $\mathcal{C}$. Define $\mathcal{A} + \mathcal{B}$ and $\mathcal{A}\mathcal{B}$ as subsets of $\mathcal{C}$ in the obvious way. Assume that

$$\mathcal{C} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \cap \mathcal{B} = \{0\} \quad \text{and} \quad \mathcal{A}\mathcal{B} = \{0\}.$$

(One often says that $\mathcal{C}$ is the internal direct sum of $\mathcal{A}$ and $\mathcal{B}$ when these conditions are satisfied.) Show that $\mathcal{A}$ and $\mathcal{B}$ are (two-sided) ideals in $\mathcal{C}$ and that $\mathcal{C}$ is $*$-isomorphic to $\mathcal{A} \oplus \mathcal{B}$.

b) Let $\mathcal{C}$ be a $C^*$-algebra and $\mathcal{A}$, $\mathcal{B}$ be $C^*$-subalgebras of $\mathcal{C}$. Assume that

$$\mathcal{C} = \mathcal{A} + \mathcal{B} \quad \text{and} \quad \mathcal{A}\mathcal{B} = \{0\}.$$

Show that $\mathcal{A}$ and $\mathcal{B}$ are (closed) ideals in $\mathcal{C}$ and that $\mathcal{C}$ is $*$-isomorphic to $\mathcal{A} \oplus \mathcal{B}$.

c) Let $\mathcal{A}$ be a unital $C^*$-algebra with unit $e$. Define $\widetilde{\mathcal{A}}$ as a $*$-algebra in the same way as we did when $\mathcal{A}$ was nonunital.
Show that $\widetilde{\mathcal{A}}$ is the internal direct sum of $\mathcal{A}'$ and $\mathcal{B}$, where

$$\mathcal{A}' := \mathcal{A} \times \{0\} \quad \text{and} \quad \mathcal{B} := \mathbb{C}(I - e) = \{\lambda\,(-e, 1) : \lambda \in \mathbb{C}\}.$$

Deduce that $\widetilde{\mathcal{A}}$ is $*$-isomorphic to $\mathcal{A} \oplus \mathbb{C}$. (Hence we may transport the $C^*$-norm on $\mathcal{A} \oplus \mathbb{C}$ to a $C^*$-norm on $\widetilde{\mathcal{A}}$, so that $\widetilde{\mathcal{A}}$ becomes a $C^*$-algebra also when $\mathcal{A}$ is unital. However one should be aware that many authors prefer just to set $\widetilde{\mathcal{A}} = \mathcal{A}$ in this case.)