MAT4360 - Fall 2017 - Exercises for Monday 30.10

Exercise 24.

Let H be a Hilbert space and let φ be a linear functional on $\mathcal{B}(H)$.

a) Show that the following conditions are equivalent:

(i) There exist $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in H$ such that

$$\varphi(T) = \sum_{j=1}^{n} \langle T\xi_j \, , \, \eta_j \rangle \quad \text{for all } T \in \mathcal{B}(H)\,,$$

(ii) φ is WOT-continuous,

(iii) φ is SOT-continuous.

(Of course, any of these conditions implies that φ is norm-continuous).

b) Deduce that if $\mathcal{C}$ is a convex subset of $\mathcal{B}(H)$, then $\overline{\mathcal{C}}^{\text{SOT}} = \overline{\mathcal{C}}^{\text{WOT}}$.

Exercise 25. (NB: This exercise will be a part of the compulsory assignment this fall).

Let Ω be a compact Hausdorff space and let μ be a finite regular Borel measure on Ω. It will be helpful to be somewhat pedantic and consider elements of the Hilbert space $\mathcal{L}^2(\mu)$ as equivalence classes of the form $[\xi]$ where ξ is any Borel measurable complex function on Ω such that $\int_\Omega |\xi|^2 \, \mathrm{d}\mu < \infty$ (so $[\xi]$ is the set of all Borel measurable complex functions on Ω agreeing with ξ μ-almost everywhere). The inner product on $\mathcal{L}^2(\mu)$ is then given by

$$\langle [\xi], [\eta] \rangle = \int_\Omega \xi \, \overline{\eta} \, \mathrm{d}\mu \quad \text{for all } [\xi], [\eta] \in \mathcal{L}^2(\mu)\,.$$

Set $\mathcal{A} = C(\Omega)$ and let φ denote the positive linear functional on $\mathcal{A}$ given by

$$\varphi(f) = \int_\Omega f \, \mathrm{d}\mu \quad \text{for all } f \in \mathcal{A}\,.$$

Moreover, set $\mathcal{N} = \{f \in \mathcal{A} \mid \varphi(f^* f) = 0\}$ and $S = \text{supp}(\mu)$ (as defined in Exercise 14), i.e.,

$$S = \{\omega \in \Omega \mid \mu(U) > 0 \text{ for every open neigbourhood } U \text{ of } \omega\}.$$

Let $(H_\varphi, \pi_\varphi, \xi_\varphi)$ denote the GNS-triple associated with φ.

a) Use that $\{[f] \mid f \in \mathcal{A}\}$ is dense in $\mathcal{L}^2(\mu)$ (as known from MAT4410) to show that there exists a unitary operator $U : H_\varphi \to \mathcal{L}^2(\mu)$ such that

$$U\xi_\varphi = [1_\Omega] \quad \text{and} \quad \big[U \, \pi_\varphi(f) \, U^*\big]\big([\xi]\big) = [f \, \xi]$$

for all $f \in \mathcal{A}$, $\xi \in \mathcal{L}^2(\mu)$.

Comment: This shows that π_φ, up to unitary equivalence, is the representation of $\mathcal{A}$ on $\mathcal{L}^2(\mu)$ by multiplication operators.

b) Verify that $\ker(\pi_\varphi) = \mathcal{N} = \{f \in \mathcal{A} \mid f = 0 \text{ on } S\}$. Then show that the C^*-algebra $\pi_\varphi(\mathcal{A})$ is $*$-isomorphic to $C(S)$.

NB: One more exercise on the next page !

Exercise 26. (NB: This exercise will be the last one in the compulsory assignment this fall).

a) Let φ be a positive linear functional on a unital C^*-algebra $\mathcal{A}$ and let $B \in \mathcal{A}$ be *invertible.* Define $\psi : \mathcal{A} \to \mathbb{C}$ by $\psi(A) = \varphi(B^*AB)$ for all $A \in \mathcal{A}$.

Check that ψ is also a positive linear functional on $\mathcal{A}$. Then show that π_ψ is unitarily equivalent to π_φ.

b) Set $\mathcal{A} = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ and let ψ be any *faithful* positive linear functional on $\mathcal{A}$. Show that π_ψ is unitarily equivalent to π_{Tr}.

Here, Tr denotes the canonical trace-functional on $\mathcal{A}$, cf. Exercise 17.

Comment: As we have seen in a lecture, H_{Tr} is equal to $\mathcal{A}$, considered as a Hilbert space w.r.t. $\langle A, B\rangle = \mathrm{Tr}(B^*A)$, and π_{Tr} is nothing but the representation of $\mathcal{A}$ on itself by left multiplication.