UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT3400/4400 — Linear Analysis with Applications
Day of examination:	Wednesday, 2 June 2021
Examination hours:	15.00 – 19.00

This problem set consists of 2 pages.

Appendices:	None.
Permitted aids:	All.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subproblems count equally. If there is a subproblem you cannot solve, you may still use the result in the sequel. All answers have to be substantiated.

Problem 1 (weight 20%)

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and define the function H on $\overline{\mathbb{R}}$ by

$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

Let $\mathcal{I}$ denote the collection of all open intervals (a, b) for $a, b \in \overline{\mathbb{R}}$ (with the usual convention $(a, a) = \emptyset$) and define the function $\varrho\colon \mathcal{I} \to \mathbb{R}$ by

$$\varrho((a, b)) = H(b) - H(a).$$

Let μ^* be the outer measure generated by $\mathcal{I}$ and ϱ.

(a) Prove that

$$\mu^*(A) = \begin{cases} 1, & \text{if } 0 \in A, \\ 0, & \text{if } 0 \notin A, \end{cases}$$

for every $A \subseteq \mathbb{R}$.

(b) Prove that every set $E \subseteq \mathbb{R}$ is μ^*-measurable.

Problem 2 (weight 40%)

Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose that there is a set $A \in \mathcal{A}$ such that $0 < \mu(A) < \infty$. Let $H = L^2(X, \mathcal{A}, \mu)$ be the real Hilbert space of real-valued functions f on X. Consider the set

$$N = \{f \in H \,:\, f(x) \geq 0 \text{ for almost every } x \in X\}.$$

(Continued on page 2.)

(a) Prove that N is not a subspace of H.

(b) Prove that the smallest subspace (of H) which contains N is H.

(c) Prove that N is closed in H.

(d) Suppose that g is a fixed element in H and define

$$d_H(g, N) = \inf_{f \in N} \|f - g\|.$$

Is there a unique element $f \in N$ such that $d(g, N) = \|f - g\|$?

Problem 3 (weight 10%)

Let H be a Hilbert space and let T be a bounded linear operator on H. Prove that

$$\ker(T^*T) = \ker(T).$$

Hint. Prove that $\ker(T) \subseteq \ker(T^*T)$ and that $\ker(T^*T) \subseteq \ker(T)$.

Problem 4 (weight 30%)

Let H be an infinite dimensional Hilbert space.

(a) Let n be a positive integer. Suppose that $\operatorname{rank}(S) = n - 1$ and that $E_n = \{e_1, e_2, \dots, e_n\}$ is an orthonormal set in H. Prove that

$$\ker(S) \cap \operatorname{span}(E_n) \neq \{0\}.$$

Hint. Consider the restriction of S to $X = \operatorname{span}(E_n)$.

Let T be a compact self-adjoint operator on H and assume that 0 is not an eigenvalue of T. Let $\{\lambda_k\}_{k \geq 1}$ denote the sequence of eigenvalues of T repeated according to their multiplicity and enumerated decreasingly (so that $|\lambda_k| \geq |\lambda_{k+1}|$ for every $k \geq 1$). Define

$$a_n(T) = \inf_{\substack{S \in \mathcal{B}(H) \\ \operatorname{rank}(S) = n-1}} \|T - S\|$$

for positive integers n.

(b) Prove that $a_n(T) \leq |\lambda_n|$ for every $n \geq 1$.

(c) Prove that $a_n(T) \geq |\lambda_n|$ for every $n \geq 1$. *Hint.* Use (a).

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