MAT3400/4400 - Linear Analysis with Applications, Autumn 2014
Reports from the lectures
Nadia Larsen

Week 34
Lecture 1. Short presentation of the course. From McDonald and Weiss I covered the following: in section 3.1, the definition and properties of the Borel measurable functions (we defined them to be the smallest collection, denoted $\widehat{\mathcal{C}}$, of real-valued functions on $\mathbb{R}$ that contains the continuous functions and is closed under pointwise limits), definition of Borel sets, some basic properties of Borel measurable functions; in section 1.4, definitions and basic examples of an algebra and a σ-algebra of subsets. The proofs of propositions 1.14 and 1.15 about existence of the smallest algebra (or σ-algebra) containing a given collection of subsets of a fixed set are left as exercises for next group session. The material on monotone classes and disjointizing a countable union of sets will be covered later. Finally, I proved that the Borel sets form a σ-algebra $\mathcal{B}$ of subsets of $\mathbb{R}$ (section 3.1). Note: in McDonald and Weiss the notation for the real numbers is $\mathcal{R}$. Either notation, or just R, is fine.
Lecture 2. I proved that every open set is a countable disjoint union of open intervals (Proposition 2.13). This implies that all open sets are Borel sets (an exercise for the group session). In section 3.2, I introduced another collection of functions, $\mathcal{F}$, and proved that it coincides with the collection of Borel measurable functions. From this we deduced the result that $\mathcal{B}$ is the smallest σ-algebra of subsets of $\mathbb{R}$ that contains all the open sets (Thm. 3.3). Note that in many books, what we here denoted $\mathcal{F}$ is taken as the definition of the Borel measurable functions on $\mathbb{R}$, and $\mathcal{B}$ is defined to be the smallest σ-algebra of subsets of $\mathbb{R}$ that contains all the open sets. From now on, either $\widehat{\mathcal{C}}$ or $\mathcal{F}$ can be used as the definition of the Borel measurable functions on $\mathbb{R}$.

Week 35
Lecture 3. Section 3.2. I introduced Lebesgue outer measure λ^* defined for all subsets of $\mathbb{R}$ and proved basic properties: that it is nonnegative, monotone, countably subadditive, takes $\emptyset$ to 0, and equals length for all intervals. I introduced the Carathéodory condition for a subset E of $\mathbb{R}$ and proved, using just this definition and the definition and properties of λ^*, that all intervals of the form (a, ∞) for $a \in \mathbb{R}$ satisfy it.
Lecture 4. Section 3.4. I introduced the collection $\mathcal{M}$ of subsets of $\mathbb{R}$ that satisfy the Carathéodory condition. These are called the Lebesgue measurable sets. I proved that $\mathcal{M}$ is a σ-algebra which contains the Borel σ-algebra $\mathcal{B}$ (for this, we proved that $\mathcal{M}$ contains all open intervals.) I showed that the restriction of λ^* to $\mathcal{M}$ is countably additive: this restriction, denoted λ, is the *Lebesgue measure on* $\mathcal{M}$. We proved basic properties of λ, such as continuity and outer regularity (exercise 3.42).

Week 36

Lecture 5. Section 4.1. Erik Bédos went through the construction of the Lebesgue integral for nonnegative functions. Note that all functions involved must be measurable before we can introduce the integral. For the purposes of chapter 4 we work with Lebesgue measurable functions. The first important concept is that of a simple function. The crucial result that makes the construction possible (and useful) is the approximation of a nonnegative $\mathcal{M}$-measurable function f by a nondecreasing sequence of nonnegative simple functions that converges pointwise to f; this is Prop. 4.3.

Lecture 6. Section 4.2. Bas Jordans went through properties of the Lebesgue integral for nonnegative functions. He proved the monotone convergence theorem (MCT), Theorem 4.3. One consequence of this result is linearity of the Lebesgue integral for nonnegative functions, Prop. 4.5. Another important result that relates a pointwise converging sequence of functions and the Lebesgue integral is Fatou's Lemma, see Theorem 4.6.

Week 37

Lecture 7. Section 4.3. I went through the construction of the Lebesgue integral for an arbitrary $\mathcal{M}$-measurable function $f : \mathbb{R} \to \mathbb{R}$. The important ingredient is the reduction to the case of nonnegative functions by decomposing f as a difference of two nonnegative functions, its positive part and its negative part. I proved basic properties of the integral including linearity, and finished the lecture proving the second major result of the theory, the dominated convergence theorem (DCT), Theorem 4.8.

Lecture 8. Sections 4.3 and 4.4. I went through some corollaries of DCT: one which relates summability of a series of functions and the corresponding series of integrals (Corollary 4.2), the other expressing a sort of continuity for the integral (Corollary 4.3). I introduced the concept of Lebesgue almost everywhere (λ-ae.) and proved basic results. These lead to a definition of the Lebesgue integral of a function f that is only defined "almost everywhere" (meaning that $f(x)$ makes sense except possibly a set of Lebesgue measure zero), see Def. 4.9. I finished the lecture by proving that every Riemann integrable function on a bounded interval $[a, b]$ is Lebesgue integrable and, moreover, in this case the integrals coincide. I presented a slightly different proof than the book: the point is to use the information on Riemann integrability to extract a nondecreasing sequence of step functions $\phi_n \leq f$ and a nonincreasing sequence of step functions $f \leq \psi_n$ by choosing a sequence of partitions through succesive refinement. Then $\psi = \inf\{\psi_n\}$ and $\phi = \sup\{\phi_n\}$ are measurable, and satisfy $\phi \leq f \leq \psi$ and

$$\int_{[a,b]} \phi \, d\lambda = \int_a^b f(x)dx = \int_{[a,b]} \psi \, d\lambda.$$

These equalities imply, since $\psi - \phi \geq 0$, that $\int_{[a,b]}(\psi - \phi)\, d\lambda = 0$. Then $\phi = \psi$ λ-ae by exercise 4.57 (the next group session). We conclude that also $f = \phi$ λ-ae and that the Lebesgue integral of f coincides with that of ϕ and ψ (by Propositions 4.7 and 4.9), hence with the Riemann integral of f.

Week 38
Lecture 9. I went through section 5.1 on abstract measure spaces $(\Omega, \mathcal{A}, \mu)$, examples of measures and properties of measures. I started on section 5.2 on $\mathbb{R}$-valued $\mathcal{A}$-measurable functions and their properties (including Thm 5.3).
Lecture 10. Section 5.2: I introduced extended real-valued functions, discussed measurability for such functions, and properties (Theorem 5.5). I defined the abstract Lebesgue integral for simple nonnegative $\mathbb{R}$-valued functions and for $\mathcal{A}$-measurable functions $f : \Omega \to [0, \infty]$ (section 5.3). I presented properties of the integral and proved the monotone convergence theorem (Thm 5.6), and also a μ-ae version. I stated Fatous' Lemma (Thm 5.7).

Week 39
Lecture 11. Sections 5.3 and 5.4: I introduced the concept of measurability for $\mathbb{C}$-valued functions and I defined the Lebesgue integral for $\overline{\mathbb{R}}$-valued and $\mathbb{C}$-valued measurable functions. I proved Propositions 5.9, 5.10, and parts of Thm 5.8.
Lecture 12. I proved the dominated convergence theorem (thm 5.9) under the assumption that the given sequence $\{f_n\}_n$ convergences μ-ae to an $\mathcal{A}$-measurable function. As an auxiliary result I proved that for $\mathcal{A}$-measurable functions (extended-real valued) $f = g$ μ-ae, integrability of one function implies integrability of the other and in that case the integrals are equal (this is based on Problem 5.52 presented earlier). I started on the preliminaries needed to introduce the product of two measure spaces using Dynkin systems (see lecture notes of September 24). More precisely, I proved Dynkin's result and a theorem about uniqueness of measures. This works for σ-finite spaces where, by definition, a measure space $(X, \mathcal{A}, \mu)$ is called σ-finite if there is a sequence $\{X_n\}_n$ of $\mathcal{A}$-measurable sets with $X_n \subseteq X_{n+1}$ for all n, $X = \cup_n X_n$, and $\mu(X_n) < \infty$ for all $n \geq 1$.

Week 40
Lecture 13. S. Neshveyev presented the construction of the product measure of two sigma-finite measure spaces (existence and uniqueness). He proved the main theorems about change of order of integration: Tonelli's and Fubini's theorems. See the lecture notes and section 6.4 in McDonald & Weiss.
Lecture 14. E. Bedos introduced the $\mathcal{L}^p$ spaces for $1 \leq p \leq \infty$, section 13.4 in McDonald & Weiss. One then defines the space $L^p(\Omega, \mathcal{A}, \mu)$ by making the convention that functions in $\mathcal{L}^p(\Omega, \mathcal{A}, \mu)$ that are equal μ-ae represent the same element. In the remaining of the lecture Erik proved the theorems of Hölder and Minkowski, the last one being the triangle inequality for the p-norm.

Week 41
Lecture 15. I went through the Riesz-Fischer theorem (Theorem 13.11) which asserts that the $\mathcal{L}^p$-spaces are Banach spaces. I presented the proof in the case $1 \leq p < \infty$. The auxiliary result that was crucial here is a result valid for an arbitrary normed space. This result says that a normed space X is a Banach space if and only if every series $\sum_{n\geq 1} f_n$ in X is convergent if it is absolutely convergent (meaning the series with positive terms $\sum_{n\geq 1} \|f_n\|$ is convergent, see for example Proposition 13.3

in McDonald & Weiss.) In the remaining of the lecture I started presenting the construction of the Cantor set (section 2.4).
Lecture 16. I continued the discussion about the Cantor set, and showed that it is a closed, uncountable set with Lebesgue measure zero. The crucial fact is that the Cantor set consists of all numbers in the interval $[0, 1]$ that have a ternary (or base-3) expansion without the digit 1. I constructed the Cantor function, and based on the knowledge that it is nonincreasing (proof left as exercise) proved that it is continuous. In the second hour I started on section 2.1 in Rynne and Youngson on abstract properties of normed spaces. Here I proved that a finite dimensional vector space admits a "standard norm" defined similar to the norm on $\mathbb{R}^n$ or $\mathbb{C}^n$.

Week 42
Lecture 17. I discussed equivalence of norms on vector spaces and proved that on a finite-dimensional vector space all norms are equivalent, see section 2.2 in Rynne and Youngson. I discussed examples of normed spaces, and I started on the theory of Banach spaces, section 2.3. We proved Thm 2.26 showing that the unit-ball and unit sphere in an infinite dimensional normed spaces are not compact.
Lecture. This lecture was cancelled because I was sick. We will compensate for it with an extra lecture in early November.

Week 43
Lecture 18. I started discussing inner-product spaces, section 3.1. We proved the fundamental inequality, the Cauchy-Schwarz inequality.
Lecture 19. I discussed orthogonality of vectors, the orthogonal complement of a set, and properties of the orthogonal complement of a set in an inner product space, sections 3.2 and 3.3. We started on the theory of Hilbert spaces (Def. 3.23). In particular, I proved Thm 3.32 about existence of a unique vector that attains the distance to a convex set in a Hilbert space. We will discuss the Gram-Schmidt orthonormalisation procedure in connection with orthonormal bases in section 3.4

Week 44
Lecture 20. I proved the result about orthogonal decomposition with respect to a closed subspace in a Hilbert space H, Theorem 3.34. As corollaries, the double orthogonal complement of a (closed) subspace Y of H was shown to be Y respectively $\overline{Y}$. We started on section 3.4 about orthonormal sequences in a Hilbert space. Here we proved Bessel's inequality (Lemma 3.41), existence of an orthonormal-sequence in an infinite-dimensional inner-product space (Thm 3.40), and summability of series of the form $\sum_n \alpha_n e_n$ for $\{e_n\}_n$ an orthonormal sequence (Theorem 3.42, Corollary 3.44).
Lecture 21. We proved Theorem 3.47 about an orthonormal sequence $\{e_n\}_n$ in H whose orthogonal complement is the trivial subspace $\{0\}$: we called such $\{e_n\}_n$ an orthonormal basis for H. I stated Thm 3.52 saying that an infinite-dimensional Hilbert space has an orthonormal basis if and only if it is separable (you can read the proof in the book). I started on section 4.1: here I discussed equivalent formulations of

continuity for a linear map between normed spaces, and we looked at examples (up to 4.4).

Week 45

Lecture 22. Section 3.5: I discussed the construction of orthonormal bases in $L^2[0, \pi]$ and $L^2[-\pi, \pi]$, viewed both as real-vector spaces and as complex vector-spaces. I finished section 4.1.

Lecture 23. Section 4.2: the norm of a bounded linear operator. Section 4.2: I proved that the space $B(X, Y)$ of bounded linear operators from a normed space X to a Banach space Y is again a Banach space for the norm defined in Lemma 4.15.

Lecture 24. I finished section 4.1 about the algebra structure of $B(X)$ (notation for $B(X, X)$). From section 5.1 I proved The Riész-Frechet theorem (Thm 5.2) describing all bounded linear functionals on a complex Hilbert space. Section 6.1: I proved existence of the dual operator T^* associated to any $T \in B(H)$, where H is a complex Hilbert space, and I discussed properties of the dual, up to Lemma 6.11.

Week 46

Lecture 25. Section 6.2: Self-adjoint and unitary operators (starting from Def 6.21.). Section 6.4: positive elements and projections. Theorem 6.51 about the characterisation of an orthogonal projection P in $B(H)$ in terms of a closed subspace of H (the subspace is $P(H)$, or, using the notation in the book, $\mathrm{Im}(P)$).

Lecture 26. Section 6.2: I finished the proof of Thm 6.51, and proved Lemmas 6.52 and 6.53. Section 7.1: compact operators. We proved properties of compact operators, and the main theorem, Thm 7.9, which says that the set of compact operators $K(X, Y)$ between a normed space X and a Banach space Y is a closed subset of $B(X, Y)$. We proved Thm 7.8(a) and will do part (b) in the next lecture.

Week 47

Lecture 27. Section 7.1 in Rynne and Youngson's book: we covered the material up to example 7.11. Section 7.2: eigenvalues and properties of these for self-adjoint and for compact operators: Thm 7.19, Thm 7.22, Cor 7.23.

Lecture 28. Section 7.3: Theorem 7.32 (we followed the proof of Lemma 14.1 in McDonald and Weiss' book). This material is also covered in section 14.3 of McDonald and Weiss' book. We finished by going through example 14.9 in McDonald and Weiss (assume $\mu(\Omega) < \infty$, then Fubini's theorem will apply.)