Extra Exercise 11

Consider $\{a_{(n,k)}\}_{(n,k)\in\mathbb{N}\times\mathbb{N}} \subseteq \overline{\mathbb{R}}_+$,

$$S = \sum_{(n,k)\in\mathbb{N}\times\mathbb{N}} a_{(n,k)} = \sup\left\{ \sum_{(n,k)\in F} a_{(n,k)} \;\middle|\; F\subseteq \mathbb{N}\times\mathbb{N},\ F \text{ finite} \right\},$$

$\sigma : \mathbb{N}\to\mathbb{N}\times\mathbb{N}$ bijection, $b_m = a_{\sigma(m)}$ for each $m\in\mathbb{N}$.

a) $$\boxed{\sum_{m=1}^{\infty} b_m \underset{(1)}{=} S \underset{(2)}{=} \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{(n,k)}\right)}$$

(1):

- Let $M\in\mathbb{N}$. Set $F_M = \{\sigma(1),\dots,\sigma(M)\}$. Then $\sum_{m=1}^{M} b_m = \sum_{(n,k)\in F_M} a_{(n,k)} \le S$.

  So $\sum_{m=1}^{\infty} b_m = \lim_{M\to\infty} \sum_{m=1}^{M} b_m \le S$.

- Let $F\subseteq\mathbb{N}\times\mathbb{N}$, F finite. Set $M_F = \max(\sigma^{-1}(F))\in\mathbb{N}$.

  Then $\sum_{(n,k)\in F} a_{(n,k)} = \sum_{m\in\sigma^{-1}(F)} b_m \le \sum_{m=1}^{M_F} b_m \le \sum_{m=1}^{\infty} b_m$

  So $S \le \sum_{m=1}^{\infty} b_m$

(2):

- Let $N, K\in\mathbb{N}$. Then $\sum_{n=1}^{N}\left(\sum_{k=1}^{K} a_{(n,k)}\right) = \sum_{(n,k)\in\{1,\dots,N\}\times\{1,\dots,K\}} a_{(n,k)} \le S$

  Letting $K\to\infty$, and next $N\to\infty$, we get $\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{(n,k)}\right) \le S$.

- Let $F\subseteq\mathbb{N}\times\mathbb{N}$, F finite. Choose $N,K\in\mathbb{N}$ so that $F\subseteq\{1,\dots,N\}\times\{1,\dots,K\}$.

  Then $\sum_{(n,k)\in F} a_{(n,k)} \le \sum_{n=1}^{N}\left(\sum_{k=1}^{K} a_{(n,k)}\right) \le \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{(n,k)}\right)$. So $S \le \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{(n,k)}\right)$

b) Assume $\{A_{(n,k)}\}_{(n,k)\in\mathbb{N}\times\mathbb{N}} \subseteq \mathcal{P}(X)$, $\rho : \mathcal{P}(X)\to[0,\infty]$, σ as above,

$B_m = A_{\sigma(m)}$ for each $m\in\mathbb{N}$.

Set $a_{(n,k)} = \rho(A_{(n,k)})$ for each (n,k). Then $\rho(B_m) = \rho(A_{\sigma(m)}) = a_{\sigma(m)}$ for each m

Using a) we get

$$\sum_{m=1}^{\infty} \rho(B_m) = \sum_{m=1}^{\infty} a_{\sigma(m)} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{(n,k)}\right) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \rho(A_{(n,k)})\right)$$

as desired.